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Extracting Probabilistic Information from the Prices of Interest Rate Options: Tests of Distributional Assumptions*

Return distributions in general and interest rates in particular have been observed to exhibit skewness and kurtosis that cannot be explained by the (log)normal distribution. Using g-and-h distribution we derived a closed-form option pricing formula for pricing European options. We measured its performance using interest rate cap data and compared it with the option prices based on the lognormal, Burr-3, Weibull, and GB2 distributions. We observed that the g-and-h distribution exhibited a high degree of accuracy in pricing options, much better than those other distributions in extracting probabilistic information from the option market.

I. Introduction

The options markets provide information on market expectations concerning the probability distribution of the underlying asset or instrument. Option pricing models can be used to infer these market expectations regarding the end-of-period probability distribution of the underlying asset or instrument and to estimate its parameter values based on observed prices, because option prices themselves are linked to pricing models featuring assumed probability distributions. We will call this an option-implied distribution. Black (1976) is a widely used model for the options we price here. Therefore, in our application the benchmark model is the Black model. The distribution assumed in the Black model is the lognormal distribution. For the instruments we will be using,[1] prices are quoted in terms of the ex ante standard deviation

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1. In the foreign exchange and interest rate markets, an implied volatility is quoted, whereas in the equity market actual price is quoted. The model used in both cases is the Black-Scholes model, which is slightly different from the Black (1976) model.

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(known as the *implied volatility*) of the lognormal distribution. Under the Black model, one obtains different implied volatilities for different strike prices even when the option expiry date is the same, giving rise to the so-called volatility smile.

Dutta and Babbel (2002) observed that 1-month and 3-month London Inter Bank Offer Rate (LIBOR) do not conform to a lognormal distribution and the skewness and kurtosis in the rates can be modeled effectively by some "flexible" leptokurtic distribution. Hawkins, Rubinstein, and Daniels (1996); Sherrick, Garcia, and Tirupattur (1996); Jondeau and Rockinger (2000); Navatte and Villa (2000); and Bhara (2001) are examples of some of the studies where the implied lognormal distributional assumption of the underlying asset or instrument was empirically rejected. After Rubinstein (1994) made a powerful argument for using general distributions in pricing the options, some efforts were made to recover the probabilistic information implied by option prices using general distributions. In this effort, GB2 (McDonald and Bookstaber 1991), Burr type 3 (hereafter, Burr-3, see Sherrick et al. 1996), and Weibull (Savickas 2001) are some of the distributions used to price options on various assets. In these and other studies, GB2 (McDonald and Bookstaber 1991) was the most general distribution and the very first attempt to price an option using a general distribution. However, only one of these attempts has been in the area of interest rate options. The closest was Rebonato (1999), who used GB2 to price a Deutschemark (DEM) cap. However, Rebonato offered no comparison of pricing accuracy with other distributional assumptions.

Guided by the evidence of the distributional properties of the 1-month and 3-month LIBOR as noted in Dutta and Babbel (2002), we price interest rate options on this instrument with a g-and-h distribution and compare it with other (lognormal, Weibull, Burr-3, and GB2) distributional assumptions in extracting the probability distribution from the option market. We consider here the European interest rate options. Our choice is influenced by the size and the nature of the products we use. Because the U.S. dollar interest rate cap is one of the most liquid interest rate options available on LIBOR, we use it to evaluate the performance of our distributional assumptions.

First, we develop the necessary framework for option pricing. Second, we price interest rate caplets under the assumption of g-and-h and other distributions. Finally, we will compare the performance of several models in extracting the implied distribution.

II. Framework for Option Prices

To price interest rate options based on different distributional assumptions, we first develop the necessary framework for option pricing. Babbel and Merrill (1996), Hull (2000), and Brigo and Mercurio (2001) are among the sources that provide comprehensive coverage on this subject.

Under general asset pricing theory with the no-arbitrage condition, the current price of an asset is equal to the present value of its expected payoffs discounted at an appropriate rate. The expectation can be made under any distributional assumption. Other than that the distribution needs to assume positive values,[2] no known economic theory can be used to justify any particular assumption. We start with Black's model and show why it is theoretically consistent with the general asset pricing theory in pricing a wide variety of European interest rate options.

Black's model calculates the expected payoff from the option assuming

1. The underlying variable X is lognormally distributed at the expiry of the option with standard deviation of $\sigma\sqrt{T}$, where T is the time to maturity.
2. The expected value of X at the maturity of the option is the forward value (F_0) of X at time 0, the valuation date.[3]

This expected payoff is then discounted by the T-duration risk-free rate at time zero. If we are pricing the call option, then the payoff from the option is $\max(X_T - c, 0)$ at time T, where X_T is the value of X at time T and c is the strike rate of the option. The price of the call option is

$$e^{-rT} \int_0^\infty \max(X_T - c, 0) f(X_T) d(X_T), \tag{1}$$

where $f(\cdot)$ is the density function of the lognormal distribution in Black's model. By evaluating the integral in (1), we obtain[4]

$$e^{-rT}[F_0 N(d_1) - cN(d_2)], \tag{2}$$

where $N(\cdot)$ is the cumulative distribution function of the standard normal, $E(X_T) = F_0$, $d_1 = [\ln(F_0/c) + \sigma^2 T/2]/\sigma\sqrt{T}$, and $d_2 = d_1 - \sigma\sqrt{T}$.

In this derivation we assume that the interest rate is either constant or deterministic. Therefore, when the interest rate is stochastic, Black's model may appear to have made approximations in terms of (1) the behavior of the interest rate and (2) assuming that $E(X_T) = F_0$. In a risk-neutral world, where the interest rate is stochastic, $E(X_T) = \tilde{F}_0 \neq F_0$, where $\tilde{F}_0$ is the future price of X at time 0. However, as shown in chapters 19 and 20 of Hull (2000), using the equivalent martingale measure in a world that is forward risk neutral with respect to a zero-coupon bond maturing at time T, the two approximations have precisely offsetting

2. If the underlying instrument is return on an asset, then the distribution can assume negative values. Some exotic options are written on asset returns.
3. The valuation date and time 0 are used interchangeably.
4. Chapter 20 in Hull (2000) gives the computation for the integral.

effects when Black's model is applied to value bond options, interest rate caps/floors, and swaptions. Therefore, when valuing these instruments, Black's model indeed has a strong theoretical basis and ensures arbitrage-free pricing.

The lognormal density assumption in Black's model is an arbitrary assumption. On the contrary, strong evidence shows that the underlying asset or instrument is often not lognormally distributed. Therefore, we replace the lognormal assumption in Black's approach with various distributions and test our assumptions. We use interest rate caplets to estimate and evaluate the parameters.

The interest rate cap is one of the most liquid interest rate options traded in the market.It comprises a portfolio of caplets. A caplet is an interest rate call option on short-term interest rates whose strike rate is the cap rate. A caplet's payoff is based on the level of the reference interest rate on the date of the caplet's maturity, but the payment is generally made in arrears.[5] The price of the cap is equal to sum of the prices of the caplets.

Suppose a cap is written on the principal amount P (known also as *notional*), with strike rate c, and for a total duration of time T (known as *tenor*). Let the entire time period T be partitioned into $t_0, t_1, t_2, \ldots, t_n$, $t_{n+1} = T$. The intermediate points in the partition are known as the reset dates. Suppose the interest rates on $t_0, t_1, t_2, \ldots, t_n$ are $r_1, r_2, \ldots, r_n$. The term $r_i\,(1 \leq i \leq n)$ are the interest rates for the periods between t_i and $t_{i+1}(1 \leq i \leq n-1)$ observed at time t_i. The caps are priced in such a way that there is no loss or gain at time t_0. For each period t_i to t_{i+1}, there is a caplet that matures at time t_I and settles (transactions made) at time t_{i+1}. The amount transacted at t_{i+1} is $P\delta_{t_i}\max(r_i - c, 0)$, where δ_{t_i} is the compounding factor[6] for the period from t_i to t_{i+1}. Since the transaction is not made until the next period, in pricing a caplet, the discounting factor in equation (1) is adjusted accordingly. The market price of a cap (and, hence, a caplet) is quoted on a notional of one currency unit. Therefore, the price of a caplet valued at time t_0, maturing at time t_i, and settling at time t_{i+1}, on a notional value of one currency unit, is

$$e^{-rt_{i+1}}\delta_{t_i}\int_0^\infty \max(X_{t_i} - c,\ 0) f(X_{t_i}) d(X_{t_i}), \tag{3}$$

where X_{t_i} is the interest rate at t_i, $f(\cdot)$ is the density function (not necessarily lognormal) of X_{t_i}, and r is the risk-free rate for the period between t_0 and t_{i+1}. Also, as before, we assume our economy is forward risk neutral. Therefore,

$$E(X_{t_i}) = F_{t_{0i}}, \tag{4}$$

where $F_{t_{0i}}$ is the forward interest rate for the period from t_0 to t_i at time t_0.

5. Not all caps pay in arrears. See Merrill and Babbel (1996).
6. Interest rates are quoted on an annualized basis.

As we have seen, the interest rate caplets are essentially an interest rate option for the period from t_0 to t_i and one of the most liquid options on interest rates. Therefore, to either extract the probabilistic information or test the distributional assumptions of the short-term interest rate, the caplets are the best instrument. In particular, we use the U.S. dollar caplets for our analysis here. With this necessary framework developed in this section, we can price the option under the different distributional assumptions.

III. Option Pricing under Different Distributional Assumptions

Since the main objective of this work is to recover the probabilistic information of the short rate from the interest-rate option market, we first price the options under different distributional assumptions. We provide in detail the option pricing formula using the g-and-h distribution, since we know of no published work where this has been done. For other distributions, we refer to and adopt the pricing formulae given in published works. A more-generalized treatment of valuation and risk management techniques across many markets and instruments is provided in Dutta (2002).

A. Option Pricing with g-and-h Distribution

The g-and-h distribution was introduced by Tukey (1977). Martinez and Iglewicz (1984), Hoaglin (1985*a*, *b*), Badrinath and Chatterjee (1988, 1991), Mills (1995), and Dutta and Babbel (2002) also studied the properties of this distribution.Badrinath and Chatterjee and Mills used the g-and-h distribution to model the return on equity indices in various markets, whereas Dutta and Babbel used it to model LIBOR rates. Tukey introduced a family of distributions by transforming the standard normal variable Z to

$$Y_{g,h}(Z) = (e^{gZ} - 1)\frac{\exp(hZ^2/2)}{g},$$

where g and h are any real numbers. By introducing location (A) and scale (B) parameters, the g-and-h distribution has four parameters in the following form:

$$X_{g,h}(Z) = A + B(e^{gZ} - 1)\frac{\exp(hZ^2/2)}{g} = A + B_{g,h}. \tag{5}$$

When $h = 0$, the g-and-h distribution reduces to $X_{g,0}(Z) = A + B[(e^{gZ} - 1)/g]$, which is also known as the g distribution. The g parameter is responsible for the skewness of the g-and-h distribution. The g distribution exhibits skewness but no kurtosis.

Similarly when $g = 0$, the g-and-h distribution reduces to

$$X_{0,h}(Z) = A + BZ\exp(hZ^2/2) = A + BY_{0,h}, \tag{6}$$

which is also known as the h distribution. The h parameter in g-and-h distribution is responsible for its kurtosis. The h distribution has fat tails (kurtosis) but no skewness. As noted in Martinez and Iglewicz (1984), many commonly used distributions can be derived as a special case of the g-and-h distribution.

To price the call option using a g-and-h distribution, we need to evaluate the integral in step (1) with the g-and-h density in (5). The integral in step (1) is equivalent to $E\{\max[(X_r - c), 0]\}$. If X_r follows a g-and-h distribution, then $X_T = a + [b(e^{gZ} - 1)e^{hZ^2/2}/g]$, where Z is a standard normal distribution. Therefore,

$$E\{\max[(X_T - c), 0]\} = \frac{1}{\sqrt{2\pi}}\int_c^\infty \left(a + \frac{b(e^{gZ} - 1)e^{hZ^2/2}}{g} - c\right)e^{-Z^2/2}dZ. \tag{7}$$

Equation (7) can be split into three parts as follows.

1. $$\frac{1}{\sqrt{2\pi}}\int_c^\infty (a - c)e^{-Z^2/2}dZ = (a - c)[1 - N(c)]. \tag{8}$$

2. $$\frac{-1}{\sqrt{2\pi}}\int_c^\infty b\frac{e^{-(1-h)Z^2/2}}{g}dZ. \tag{9}$$

Transforming $Z \to y/\sqrt{1-h}$, the integral in (9) becomes

$$\frac{1}{\sqrt{2\pi}}\int_{c(\sqrt{1-h})}^\infty b\frac{e^{-y^2/2}}{g(\sqrt{1-h})}dy = \frac{-b}{g(\sqrt{1-h})}\left[1 - N(c\sqrt{1-h})\right]. \tag{10}$$

3. $$\frac{b}{g\sqrt{2\pi}}\int_c^\infty e^{-\{[(1-h)Z^2]/2 - gZ\}}dZ. \tag{11}$$

Completing the square in the exponent of e in (11), we have

$$\frac{b}{g\sqrt{2\pi}}e^{g^2/2(1-h)}\int_c^\infty e^{-\frac{1}{2}\left[(\sqrt{1-h})Z - \frac{g}{\sqrt{1-h}}\right]^2}dZ. \tag{12}$$

By making the transformation $y \to \sqrt{1-h}Z - g/\sqrt{1-h}$ to the integral in (12), we have

$$\frac{b}{g\sqrt{1-h}} e^{g^2/2(1-h)} \frac{1}{\sqrt{2\pi}} \int_{(\sqrt{1-h})c - g/(\sqrt{1-h})}^{\infty} e^{-\frac{1}{2}(Z)^2} dZ$$

$$= \frac{b}{g\sqrt{1-h}} e^{g^2/2(1-h)} \left\{1 - N\left[\left(\sqrt{1-h}\right)c - g\Big/\left(\sqrt{1-h}\right)\right]\right\}. \quad (13)$$

Combining equations (8), (10), and (13), we get the call price using the g-and-h distribution as

$$e^{-rT}E\{\max[(X_T - c), 0]\} = e^{-rT}\Big\{(a-c)[1 - N(c)]$$
$$- \frac{b}{g(\sqrt{1-h})}\left[1 - N\left(c\sqrt{1-h}\right)\right]$$
$$+ \frac{b}{g\sqrt{1-h}} e^{g^2/2(1-h)}\left[1 - N\left(\sqrt{1-h}\right)c - g/\left(\sqrt{1-h}\right)\right]\Big\}, \quad (14)$$

where

$$E(X_T) = a + \frac{b(e^{g^2/2(1-h)} - 1)}{g\sqrt{1-h}} = F_0. \quad (15)$$

We can eliminate a parameter between equations (14) and (15) and express equation (14) in terms of F_0, the forward price of X_T on the valuation date.

Similarly we price the put option as follows:

$$E\{\max[(p - X_T), 0]\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{p} \left(p - a + \frac{b(e^{gZ} - 1)e^{hZ^2/2}}{g}\right) e^{-Z^2/2} dZ,$$

where p is the strike price for a put. Following the steps as before, we get the price of the put as

$$e^{-rT}[(p-a)N(p) + \frac{b}{g(\sqrt{1-h})} N(p\sqrt{1-h})]$$
$$- e^{-rT}\Big\{(p-a)N(p) + \frac{b}{g(\sqrt{1-h})} N(p\sqrt{1-h})$$
$$- \frac{b}{g\sqrt{1-h}} e^{g^2/2(1-h)}\left[N\left(\left(\sqrt{1-h}\right)p - g\Big/\left(\sqrt{1-h}\right)\right)\right]\Big\}. \quad (16)$$

Putting $p = c = X$ and subtracting (16) from (14), we get

$$e^{-rT}(F_0 - X). \tag{17}$$

From step (17), we conclude that our option prices using the g-and-h distribution preserve put-call parity, a necessary relationship to validate any option pricing formula. To price a caplet, we multiply the call price in (14) with δ_{t_0}, the compounding factor.

B. Option Pricing with Generalized Beta Distribution of the Second Kind (GB2)

The generalized beta distribution of the second kind (GB2), like the g-and-h distribution, can accommodate a wide variety of tail thicknesses and permits skewness as well. Bookstaber and McDonald (1987), McDonald and Bookstaber (1991), McDonald (1996), and McDonald and Xu (1995) analyzed the properties and applications of the GB2 distribution in detail. Bookstaber and McDonald (1987) and McDonald (1996) explored the possibility of modeling asset returns using GB2. GB2 distribution is defined as

$$\begin{aligned} \text{GB2}(y; a, b, p, q) &= \frac{|a| y^{ap-1}}{b^{ap} B(p, q)[1 + (y/b)^a]^{p+q}}, \quad \text{when } y > 0; \\ &= 0 \text{ otherwise.} \end{aligned} \tag{18}$$

Here, $B(p, q)$ is a beta function. Like the g-and-h, GB2 is a four-parameter distribution. Some of the useful properties of GB2 are summarized next.

The cumulative distribution function of GB2 is given by[7]

$$X(y; a, b, p, q) = z^p\, {}_2F_1 \frac{[p, 1-q, 1+p; z]}{pB(p, q)}, \tag{18a}$$

where $z = (y/b)^a/[1+(y/b)^a]$ and ${}_1F_2[a, b, c, d]$ is a hypergeometric function.[8] Bookstaber and McDonald (1987) noted that many commonly used distributions can also be derived as a special case of GB2.

McDonald and Bookstaber (1991) developed a method to price options using the GB2 distribution. The method adopted by McDonald is based on normalized incomplete moments.The hth normalized incomplete moment of a distribution is defined as $\varphi(y, h) = \left[\int_0^y s^h f(s) ds\right] / E(Y^h)$. The complement of the normalized incomplete moment is $\bar{\varphi} = 1 - \varphi$. From the density function of GB2, we can infer that computing the option prices using the expression in (1) is quite cumbersome.

7. For the derivation, see McDonald and Xu (1995) and McDonald (1996).

8. Hypergeometric is a special function. Reference for such functions is Abramowitz and Stegun (1972).

Rebonato (1999) used the method developed by McDonald and computed the call option price[9] based on GB2, which is equal to

$$e^{-rT}\left\{\begin{array}{l}\dfrac{X(b/X)^{aq}\ {}_1F_2[q-1/a,p+q,1+q-1/a,-(b/X)^a]}{(q-1/a)B(p,q)}-\\ \dfrac{X(b/X)^{aq}\ {}_1F_2[q,p+q,1+q,-(b/X)^a]}{qB(p,q)}\end{array}\right\}, \qquad (19)$$

where X is the strike rate and ${}_1F_2[j, k, l, m]$ is the hypergeometric function. Multiplying the compounding factor as before, we get the price of the caplet. The call price given in (19) is slightly different from the one given in Rebonato (1999). Rebonato assumed a zero-interest-rate economy for discounting purposes, which is not a realistic assumption. As before, the following condition should also hold to satisfy the risk neutrality of the forward prices:

$$E(X_T) = \frac{bB(p+1/a, q-1/a)}{B(p,q)} = F_0. \qquad (20)$$

Using equations (19) and (20), we can eliminate one parameter out of the four parameters of the GB2 distribution and express the call price in (19) in terms of F_0. Rebonato equated the first and second moments of the GB2 distribution with those of the lognormal and called it an *equivalent volatility*. By doing so, one can eliminate one more parameter of GB2 in the option price given in (19). There is no known empirical or economic justification for this approach. In that assumption, one would give up the flexibility of the distribution.

C. *Option Pricing with Other Distributions*

We also use Burr-3 and the Weibull distributions to compare the option prices based on g-and-h and GB2 distributions.

Burr-3 distribution. The Burr-3 distribution is a special case of the GB2 distribution (Bookstaber and McDonald 1987). When the parameter $q = 1$ in (18), we have the Burr-3 distribution:

$$\text{Burr3}(y; a, b, p) = \frac{|a|\, y^{ap-1}}{b^{ap}B(p,1)[1+(y/b)^a]^{p+1}},$$

when $y > 0$ and 0 otherwise. Therefore, Burr-3 is a distribution with three parameters. Sherrick et al. (1996) used it to price options on

9. For the price of a put. see Rebonato (1999).

soybean futures. Even though Burr-3 has received limited attention in modeling asset returns, it has been found to be useful in describing the loss distribution in the insurance industry.

Since Burr-3 is a special case of GB2 ($q = 1$), therefore substituting for the value of q in (19) and (20), we get the price of a call option (and caplet) using the Burr-3 distribution. The option price associated with the Burr-3 distribution has only two free parameters.

Weibull distribution. The Weibull distribution can also be derived as a limiting case from g-and-h as well as from GB2. The Weibull distribution belongs to the family of distributions known as *extreme-value distributions*. The density function of the Weibull distribution is given by

$$f(x) = abx^{b-1}e^{-ax^b},$$

where $a > 0$, $b > 0$, and $x > 0$. Weibull is therefore a distribution with two parameters. Savickas (2001) describes the properties of the Weibull distribution and compares it with the lognormal distribution.

Savickas developed the option price under the Weibull distribution using (1). Substituting the Weibull density in (1) we obtain the price of a call option:

$$e^{-rT}\int_c^\infty (X_T - c)\alpha\beta X_T^{\beta-1}e^{\alpha X_T^\beta}dX_T. \tag{21}$$

After simplification, equation (21) reduces to

$$e^{-rT}\left\{\frac{\Gamma(1+1/\beta)}{\alpha^{1/\beta}}\left[1-\frac{\Gamma_\omega(1+1/\beta)}{\Gamma(1+1/\beta)}\right]-ce^{-\omega}\right\}, \tag{22}$$

where $\omega = \alpha c^\beta$ and $\Gamma_\omega(1 + 1/\beta) = \int_0^\infty x^{1/\beta}e^{-x}dx$ is the incomplete gamma function (see Abramowitz and Stegun 1972). Substituting $E(X) = [\Gamma(1 + 1/\beta)]/\alpha^{1/\beta} = F_0$ in (22), we have

$$e^{-rT}\left\{F_0\left[1-\frac{\Gamma_\omega(1+1/\beta)}{\Gamma(1+1/\beta)}\right]-ce^{-\omega}\right\}. \tag{23}$$

As before, multiplying (23) by the compounding factor, we get the price of the caplet using the Weibull distribution.

We now have all the caplet prices we need to test the distributional assumptions implied by option prices. In the next section, we first estimate the parameter, then test the assumptions with the caplet data.

IV. Tests of the Distributional Assumptions

To test the distributional assumption, we first need to estimate the parameters of the distribution. The caplet prices calculated in an earlier section are used to estimate the parameters. The model price of the caplet is E^i_Δ where Δ is the set of parameters to be estimated. The term O_i is the market prices of the caplet. The term n is the total number of caplets used to estimate the parameters. Then the best estimate of Δ is $\Delta_{\min}$, the parameter set that minimizes

$$\sum_{i=1}^{n}(O_i - E^i_\Delta)^2 \tag{24}$$

subject to $E(X_T) = F_0$ and other parameter constraints specific to a particular distribution.

The optimization problem described in (24) is a nonlinear optimization problem, which adds several complexities in estimating the parameters. We encountered situations where the solution did not exist. We used the optimizer (solver)[10] in Microsoft Excel to solve the problems.

A. Data Description

According to the International Swaps and Derivatives Association, the total notional principal amount of over-the-counter U.S. dollar interest rate options, such as caps/floors and swaptions, exceeded \$6 trillion at the end of 2000. This amount was more than 50 times the \$120 billion in combined notional principal of all the options on Treasury notes and bond futures traded at the Chicago Board of Trade. Therefore, caps/floors are one of the most liquid interest rate options that can be used to infer an implied probability distribution. As we explained earlier, the liquidity of the option is important for it to be used in recovering the probability distribution.

The U.S. dollar caps are quoted in basis points.[11] The price of the contract is multiplied by the notional principal amount to give the dollar value of the contract. Since the caps are over-the-counter traded contracts, the data relating to caps are available only from the broker/dealer or market maker. Most of the available data are quoted on an at-the-money forward basis, which means that, on any trading day and for one specific tenor, the quote of only one strike is available. We noted earlier that, as we move away (in either direction) from an at-the-money forward, the option starts exhibiting the volatility smiles. The trading

10. The solver was enhanced by the Premium Solver Platform, software obtained from the Frontline Systems. We tried to solve the optimization problems by the MATLAB optimizer as well. We found that the Premium Solver Platform performed better than the MATLAB for our application.

11. The prices are normally quoted in terms of implied volatilities. However, we obtained them in basis points.

strategy of caps as well as of many other interest rate instruments is based on this volatility smile.

We obtained end-of-the-day closing prices for U.S dollar caps of different strikes and tenors from a major broker/dealer and market maker in interest rate caps/floors and swaptions. The tenors of the cap were of six different maturities (2, 3, 4, 5, 6, and 7 years) and of eight different strikes (5, 5.5, 6, 6.5, 7, 7.5, and 8%). However, caps at all these strikes were not always quoted for each of the maturities. The sample period consisted of 141 trading days[12] of daily data from October 23, 2000, to September 19, 2001. In total, 3,769 contracts were used for the estimation of the parameters. The liquidity of the contracts on a given day varied according to the strikes, maturities, and the 3-month-LIBOR rate on that day. Deep in-the-money as well as deep out-of-the-money caps exhibited less liquidity. We have disregarded any quote with an open interest less than 10. All the caps with strike 8.5% were disregarded for having open interest less than 10. Also, options with short maturities (less than 3 years) exhibited liquidity only for at-the-money-forward strikes.Therefore, even though we could obtain the data for a 1-year cap, we chose not to use it due to the lack of its liquidity. Table 1 gives the basic statistics of the cap data we used. From the cap prices we obtained the caplet prices.[13] For the purpose of computing these caplet prices we needed the forward (and discount) curve(s). We used 1-, 3-, 6-, and 12-month LIBOR, 2-, 3-, 5-, 7-, and 10-year U.S. dollar interest rate swap data to construct the forward curve (and discount curve) for each trading day. This discount curve was used in our option valuation models as well. Using these caplet prices we estimated the parameters of the implied distributions for each of the option models discussed earlier. We used the caplets that matured at the end of the ninth month and settled at the end of twelfth month from the beginning of the cap.

B. The Testing Methodology and Model Evaluation

The testing methodology we adopt here is similar to the ones in Jackwerth and Rubinstein (1995); Buhler et al. (1999); Driessen, Klaassen, and Melenbert (2000); Gupta and Subrahmanyam (2001); and Savickas (2001). Caplets were classified based on tenors. For each tenor, the parameters of the distributions were estimated using the methodology outlined in (24). The significance of this classification lies in testing the stability of the parameters and their out-of-sample performance.

Using the estimated parameters, the caplet prices were computed under each of the distributional assumptions. The (percentage) errors between the market and the model prices (both relative and absolute)

12. There were approximately 252 trading days. However, on approximately 100 trading days, there was no noticeable price movement from the previous day. In our analysis, we used distinct prices to estimate the parameters.

13. We used FINCAD tools to compute the prices.

TABLE 1 U.S. Dollar Interest Rate Caps

	Years of Tenure											
	2	3	4	5	6	7	2	3	4	5	6	7
	5% Strike						**5.5% Strike**					
Mean	54	139	235	—	—	—	36	110	201	291	381	478
Max	81	187	252	—	—	—	61	143	249	368	423	517
Min	33	120	228	—	—	—	20	79	157	241	339	433
SD	12	16	9	—	—	—	10	17	23	27	25	26
25% percentile	44	126	230	—	—	—	29	96	182	272	360	458
75% percentile	64	148	239	—	—	—	43	122	219	306	403	504
Median	54	136	231	—	—	—	36	109	197	286	379	482
Count	94	40	11	—	—	—	119	119	101	64	34	17
	6% Strike						**6.5% Strike**					
Mean	22	76	148	232	320	410	27	67	109	167	232	308
Max	57	111	194	300	404	500	53	113	173	242	303	384
Min	12	53	113	179	252	332	9	32	73	125	182	248
SD	8	14	22	30	39	43	17	29	28	30	32	35
25% percentile	16	64	127	206	288	370	13	45	87	151	217	288
75% percentile	25	85	168	258	355	441	45	93	124	180	246	318
Median	20	73	143	229	320	407	20	52	101	157	223	306
Count	124	124	124	124	123	120	44	42	34	32	31	31
	7% Strike						**7.5% Strike**					
Mean	9	36	78	132	191	253	7	25	56	99	146	202
Max	26	68	121	192	264	337	13	41	82	136	194	255
Min	2	15	44	86	131	182	2	8	27	58	94	138
SD	6	13	21	28	35	40	4	12	20	26	32	37
25% percentile	5	27	61	111	165	229	3	14	38	76	121	175
75% percentile	9	42	94	156	222	287	12	38	78	122	176	235
Median	6	32	75	131	189	250	7	28	62	98	146	194
Count	141	141	141	141	141	141	44	44	44	44	44	44
	8% Strike						**8.5% Strike**					
Mean	3	15	38	70	108	150	2	10	26	52	83	120
Max	11	27	57	99	147	195	6	18	40	72	107	50
Min	1	4	16	39	66	100	1	2	10	27	47	73
SD	2	6	12	17	23	27	1	5	11	16	21	24
25% percentile	1	10	26	56	91	132	1	5	16	38	68	102
75% percentile	3	20	49	86	130	175	3	14	35	66	101	142
Median	2	15	39	69	108	150	2	12	31	60	89	123
Count	138	141	141	141	141	141	44	44	44	44	44	44

NOTE.—This table presents the descriptive statistics of the U.S. dollar interest rate caps used to estimate the parameters of the distributions. The prices of the contracts are expressed in basis points (1 bp = 0.01%). The total number of contracts used for the estimation purpose is 3,769.

TABLE 2 **Forecast Errors (in Basis Points and Percentages) for the Lognormal, g-and-h, GB2, Burr-3 and Weibull Distributions**

		Average Error		Average Absolute Error	
Model	Tenor	(Basis Points)	%	(Basis Points)	%
Lognormal	2 yrs	3.459	−52.68	5.667	107.32
	3 yrs	2.209	−67.46	4.694	114.75
	4 yrs	−.101	−443.35	3.851	476.44
	5 yrs	−.275	−563.64	3.225	593.35
	6 yrs	−.244	−532.79	2.832	562.78
	7 yrs	−.226	−533.82	2.566	564.03
GB2	2 yrs	−.058	−12.16	.612	17.34
	3 yrs	−.050	−12.20	.507	16.90
	4 yrs	−.138	−47.23	.379	50.64
	5 yrs	−.099	−48.93	.258	51.75
	6 yrs	−.072	−41.51	.185	44.14
	7 yrs	−.056	−38.10	.136	40.54
g-and-h	2 yrs	−.038	−4.35	.189	6.07
	3 yrs	−.034	−4.63	.159	6.02
	4 yrs	−.022	−8.57	.095	9.88
	5 yrs	−.015	−6.49	.052	7.61
	6 yrs	−.011	−4.91	.034	5.71
	7 yrs	−.008	−3.70	.023	4.62
Burr3	2 yrs	4.013	−17.36	5.047	74.09
	3 yrs	3.030	−25.13	4.181	76.72
	4 yrs	5.175	99.32	5.175	99.32
	5 yrs	.404	−333.31	2.360	359.85
	6 yrs	−.018	−364.67	2.092	387.03
	7 yrs	−.336	−397.67	1.903	416.91
Weibull	2 yrs	7.033	18.91	11.478	69.52
	3 yrs	5.129	23.40	7.120	81.87
	4 yrs	8.159	−12.77	4.035	109.61
	5 yrs	−1.012	−393.39	3.699	489.46
	6 yrs	−.923	−446.07	4.647	263.00
	7 yrs	.325	−276.73	2.948	516.22

were computed. For each tenor, the average was computed by taking all caplets across different strikes and for all the trading days in our sample. Table 2 shows the average (percentage) errors (both absolute and relative) across different maturities for the option prices under various distributional assumptions.

In addition to estimating the errors, the model performance was also evaluated by the following statistical tests.

(i) The model price and market price are regressed by the following regression equation: market − price$_i = \beta_{i0} + \beta_{i1}$ + model price + ε_i, and the values of the coefficients, standard errors, and the R^2 are noted.
(ii) The correlation coefficients of the errors and percentage errors (both relative and absolute) are computed among the different models and for all maturities.
(iii) The basic statistics (mean, standard deviation, median, percentile, etc) for the parameter estimates were obtained across all maturities.

From table 2, we can see that, on the basis of both relative and absolute errors, the g-and-h distribution exhibits the highest accuracy in extracting the implied distribution. The minimum and maximum absolute average errors obtained are 4.62% and 9.88%, respectively. The average percentage relative and absolute errors from using the g-and-h distribution are significantly smaller than the errors from other distributions. Although insignificant, the negative value of the average relative error (consistently obtained across all maturities) indicates that g-and-h, on average, overpriced the options.

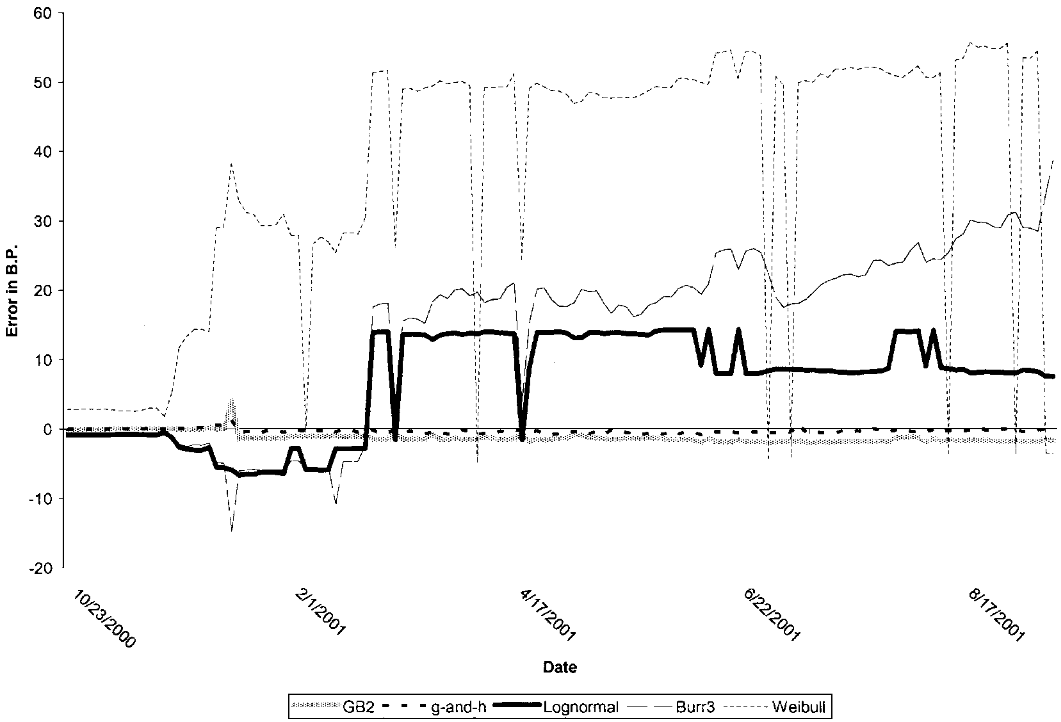
The GB2 distribution showed the next best accuracy. However, on a percentage basis, GB2 exhibited much higher inaccuracies than g-and-h. In certain instances, the solution for (24) did not exist for the GB2 distribution. Therefore, in those instances, GB2 violated arbitrage-free pricing. Like g-and-h, GB2 overpriced the options consistently but at much higher differences than g-and-h. The minimum and maximum absolute average errors were 16.9% and 51.75%, respectively. Figure 1 shows graphically the relative errors for different maturities under different distributions.

For other distributions (Burr-3, lognormal, and Weibull) the errors were extremely high, indicating that the implied distribution is different from Burr-3, lognormal, and Weibull. Also, we observed that the errors decreased from shorter maturities to longer maturities. One possible reason for this is that liquidity increased from shorter maturities to longer maturities. We had more contracts to solve (24) for the longer maturities than for the shorter maturities.

Tables 3 and 4 show the results of the regression statistic for g-and-h and GB2 distributions, respectively. From the R^2 column in the tables, we conclude that there is a high degree of correlation between market and model prices under the g-and-h distribution and in many instances under the GB2 distribution as well. While we observed no value of R^2 less than 82% for the g-and-h distribution, we observed several R^2 values for GB2 less than 60%. For the Burr-3, lognormal, and Weibull distributions, we often observed very low correlations between market and model prices (tables 5, 6, and 7).

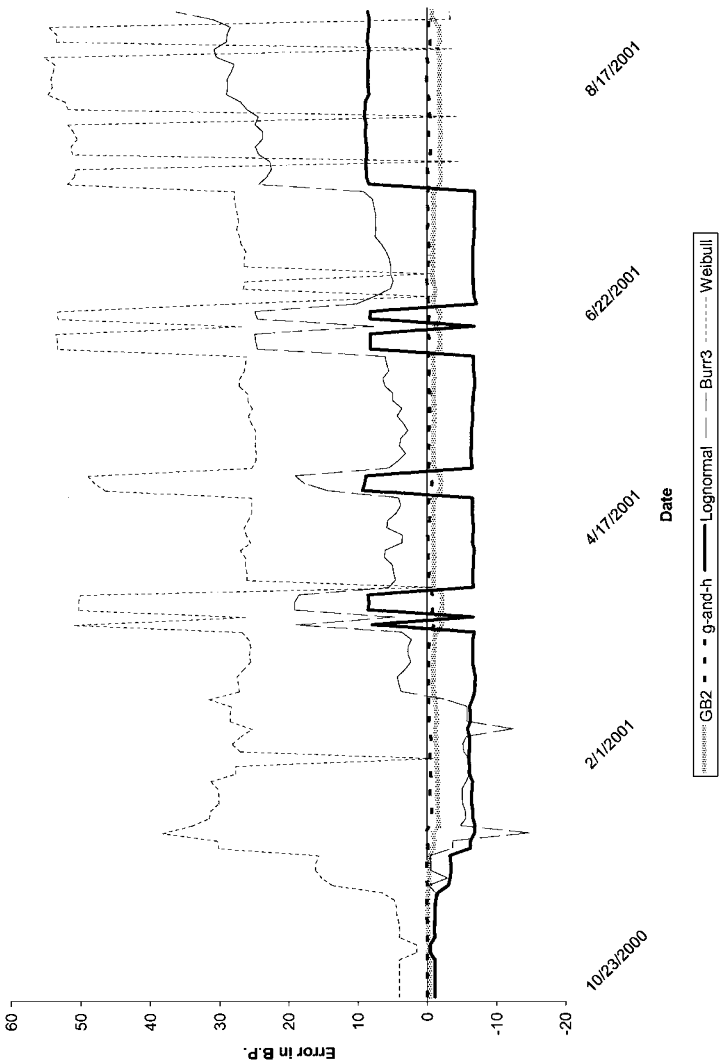
Figure 2 shows the average values of the implied parameter estimates of g-and-h for all trading days in our sample. We observed high volatilities in the parameter estimates for g-and-h. This is consistent with the g-and-h parameter estimates that Dutta and Babbel (2002) obtained using the historical 1-month and 3-month-LIBOR data. Therefore, we observe that both historical and implied estimates of the parameters of g-and-h show high volatility. Similar observations were obtained for the GB2 distribution (fig. 3).

We observed a very high degree of inaccuracy between the model and the market prices under the assumptions of Burr-3, lognormal, and Weibull distributions. Table 8 shows the correlation coefficients of errors

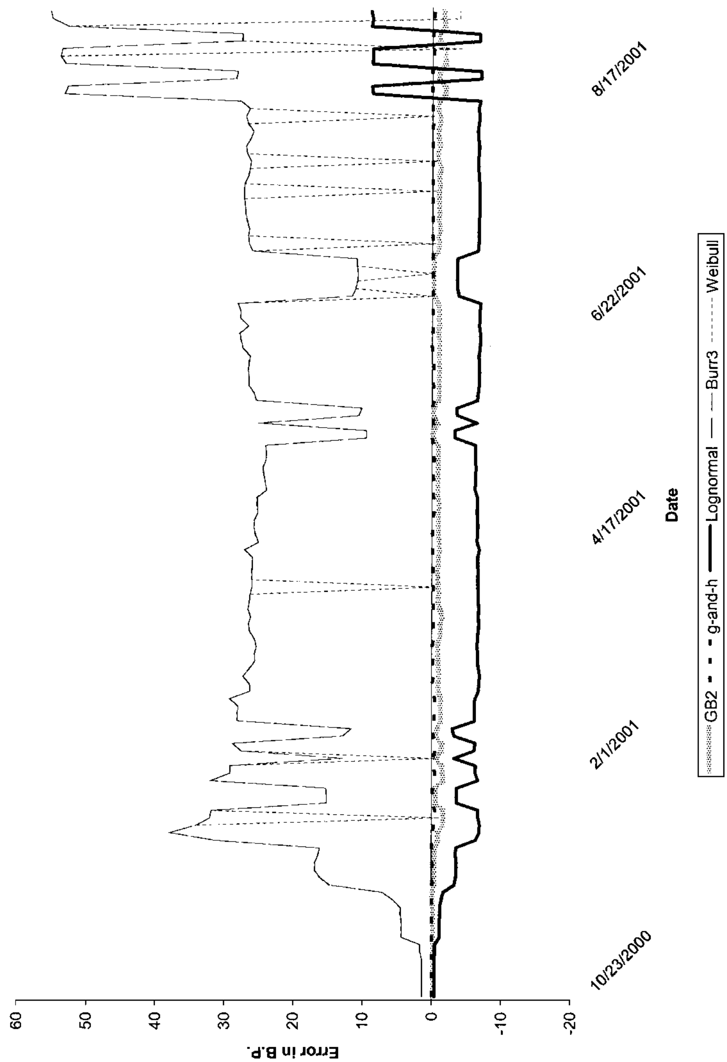


(a) 2-year maturity

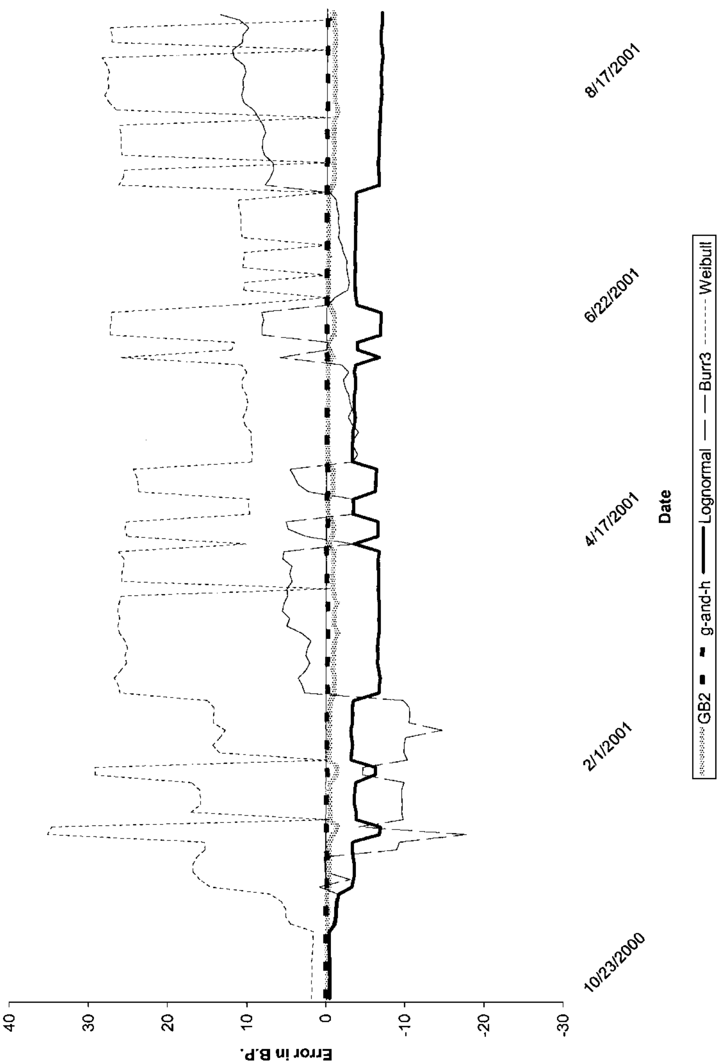
FIG. 1—Model Errors



(b) 3-year maturity



(c) 4-year maturity



(d) 5-year maturity

TABLE 3 Regression Statistics, Options with g-and h Distribution: Model vs. Market Price

Tenor	Rate (%)	*a*	*b*	R^2	SE
2-year	5.0	−.0005	.9933	.99997	.0670
	5.5	.0100	1.0158	.99871	.1983
	6.0	.0241	1.0046	.99845	.1125
	6.5	−.0007	.9995	.99959	.0316
	7.0	−.0315	.8799	.92324	.1949
	7.5	−.0007	.8941	.97227	.0490
3-year	5.0	−.0001	.9949	.99999	.0406
	5.5	−.0011	1.0013	.99912	.1641
	6.0	.0377	1.0138	.99705	.1554
	6.5	−.0013	1.0025	.99918	.0485
	7.0	.0091	.8923	.91629	.1567
	7.5	−.0036	.8745	.95948	.0631
4-year	5.0	.0000	.9954	.99998	.0261
	5.5	−.0020	.9973	.99980	.0974
	6.0	.0264	1.0124	.99738	.1426
	6.5	−.0006	1.0122	.99924	.0474
	7.0	−.0295	.9635	.95649	.0948
	7.5	−.0023	.8988	.98323	.0419
5-year	5.5	−.0002	.9976	.99997	.0417
	6.0	.0069	1.0052	.99851	.1055
	6.5	.0003	1.0093	.99971	.0293
	7.0	.0714	.9376	.95701	.0767
	7.5	−.0012	.9644	.99741	.0168
	8.0	−.0002	.8609	.82012	.0472
6-year	5.5	.0000	.9980	.99998	.0277
	6.0	.0002	1.0031	.99935	.0707
	6.5	.0003	1.0089	.99988	.0187
	7.0	.0836	.9358	.97203	.0569
	7.5	−.0009	.9755	.99847	.0130
	8.0	.0035	.8800	.88630	.0337
7-year	5.5	.0000	.9994	1.00000	.0078
	6.0	−.0005	1.0010	.99992	.0258
	6.5	−.0002	1.0085	.99990	.0172
	7.0	.0607	.9533	.97822	.0456
	7.5	−.0008	.9846	.99870	.0122
	8.0	−.0151	.9881	.95997	.0189

NOTE.—This table summarizes the regression statistics of the regression between the option prices calculated based on the model and the market price of the option using the following regression equation: Market Price = $a+b$ (Model Price) + error. The option prices are calculated under the assumption of the g-and-h distribution at the expiry of the option.

TABLE 4 **Regression Statistics, Options with GB2 Distribution: Model vs. Market Price**

Tenor	Rate (%)	*a*	*b*	R^2	SE
2-year	5.0	−.0011	.9762	.99998	.0520
	5.5	.0747	1.0505	.99085	.5285
	6.0	.0253	1.0431	.99635	.1728
	6.5	−.0071	1.0328	.99669	.0893
	7.0	.1947	.6251	.79395	.3193
	7.5	.0043	.6691	.87344	.1046
3-year	5.0	−.0004	.9777	.99999	.0374
	5.5	.0883	1.0029	.98872	.5880
	6.0	.0595	1.0610	.99332	.2337
	6.5	−.0048	1.0308	.99863	.0625
	7.0	.3483	.6246	.81655	.2320
	7.5	.0004	.6459	.88828	.1048
4-year	5.0	.0000	.9781	1.00000	.0100
	5.5	.0085	.9903	.99779	.3259
	6.0	.0579	1.0494	.99034	.2737
	6.5	−.0034	1.0476	.99824	.0723
	7.0	.4083	.6498	.82404	.1906
	7.5	.0021	.6850	.91288	.0955
5-year	5.5	−.0003	.9853	.99998	.0336
	6.0	.0445	1.0266	.98795	.3002
	6.5	−.0009	1.0552	.99916	.0499
	7.0	.5677	.6275	.75442	.1832
	7.5	.0009	.7938	.97254	.0547
	8.0	.0763	.3232	.57050	.0729
6-year	5.5	−.0002	.9854	.99998	.0254
	6.0	.0256	1.0120	.99161	.2544
	6.5	−.0008	1.0574	.99918	.0495
	7.0	.5662	.6499	.78838	.1564
	7.5	−.0033	.8466	.98917	.0347
	8.0	.0871	.3294	.57349	.0653
7-year	5.5	.0000	.9857	.99998	.0198
	6.0	.0106	1.0040	.99595	.1874
	6.5	−.0011	1.0576	.99916	.0504
	7.0	.4778	.7104	.83116	.1269
	7.5	−.0032	.8504	.99204	.0303
	8.0	.0655	.3935	.60675	.0593

NOTE.—This table summarizes the regression statistics of the regression between the option prices calculated based on the model and the market price of the option using the following regression equation: Market Price = $a + b$ (Model Price) + error. The option prices were calculated under the assumption of the GB2 distribution at the expiry of the option.

TABLE 5 Regression Statistics, Options with Burr-3 Distribution: Model vs. Market Price

Tenor	Rate (%)	a	b	R^2	SE
2-year	5.0	.6140	2.3450	.9563	2.3746
	5.5	1.8364	1.6358	.8275	2.2954
	6.0	1.6249	1.1249	.6964	1.5758
	6.5	.2012	.6965	.8834	.5302
	7.0	.9715	.2915	.1832	.6358
	7.5	.0086	.2276	.8424	.1167
3-year	5.0	.1023	2.6729	.9815	1.4858
	5.5	1.7636	1.6540	.8427	2.1964
	6.0	1.4936	1.1707	.7519	1.4244
	6.5	.2215	.7392	.8429	.6695
	7.0	1.3101	.1867	.1116	.5106
	7.5	.0242	.2247	.7290	.1633
4-year	5.0	.0033	1,040.0682	.9975	.2946
	5.5	.5670	727.1868	.9444	1.6325
	6.0	1.3738	487.2378	.7962	1.2569
	6.5	.1112	318.4974	.8229	.7253
	7.0	2.0491	−25.0283	.0123	.4517
	7.5	.0271	89.4768	.6053	.2032
5-year	5.5	.3332	1.9056	.9488	1.7394
	6.0	1.3597	1.1479	.7895	1.2550
	6.5	.1149	.7694	.8918	.5671
	7.0	1.3312	.1560	.1851	.3337
	7.5	.0385	.2451	.7862	.1527
	8.0	.1825	.0456	.0500	.1084
6-year	5.5	.1095	2.0540	.9658	1.2119
	6.0	1.2254	1.1535	.8110	1.2071
	6.5	.0982	.7744	.9025	.5389
	7.0	1.2992	.1438	.1886	.3063
	7.5	.0387	.2494	.7986	.1497
	8.0	.1602	.0475	.0686	.0965
7-year	5.5	.0331	2.2310	.9770	.7238
	6.0	.9897	1.1761	.8482	1.1472
	6.5	.0976	.7798	.9036	.5394
	7.0	1.2529	.1349	.2003	.2761
	7.5	.0375	.2565	.8156	.1456
	8.0	.1328	.0518	.0905	.0902

NOTE.—This table summarizes the regression statistics of the regression between the option prices calculated based on the model and the market price of the option using the following regression equation: Market Price = a + b (Model Price) + error. The option prices were calculated under the assumption of the Burr-3 distribution at the expiry of the option.

TABLE 6 **Regression Statistics, Options with Lognormal Distribution: Model vs. Market Price**

Tenor	Rate (%)	*a*	*b*	R^2	SE
2-year	5.0	.119954	2.716572	.994050	.875937
	5.5	1.148203	1.805696	.927341	1.489608
	6.0	1.091202	1.060303	.898276	.912176
	6.5	.169850	.826769	.903803	.481656
	7.0	.390392	.272722	.652974	.414421
	7.5	.014723	.170521	.798610	.131935
3-year	5.0	.011543	2.728297	.998902	.361917
	5.5	1.025577	1.837937	.932937	1.433926
	6.0	1.080353	1.081937	.894069	.930777
	6.5	.176353	.863081	.878630	.588423
	7.0	.753561	.208199	.565883	.356923
	7.5	.030101	.165874	.698128	.172311
4-year	5.0	.000084	2.732758	.999966	.034790
	5.5	.347975	1.927794	.971758	1.164011
	6.0	2.066477	.961362	.761322	1.360259
	6.5	.122611	.957015	.896325	.554900
	7.0	1.059690	.153940	.482502	.326924
	7.5	.036871	.187868	.716351	.172277
5-year	5.5	.045914	1.945380	.993612	.614218
	6.0	3.119459	.879656	.618824	1.688711
	6.5	.085347	1.170790	.927678	.463547
	7.0	1.306191	.117218	.407656	.284515
	7.5	.032117	.254855	.852192	.127000
	8.0	.137309	.033437	.306676	.092636
6-year	5.5	.000386	1.933716	.999950	.046274
	6.0	3.082462	.997222	.613880	1.725424
	6.5	.030777	1.357933	.975705	.269068
	7.0	1.291102	.121728	.399696	.263426
	7.5	.023082	.299831	.887799	.111704
	8.0	.142690	.031897	.265634	.085696
7-year	5.5	−.000089	1.945277	.999957	.031342
	6.0	2.163677	1.309089	.717493	1.564987
	6.5	.030656	1.359038	.975919	.269535
	7.0	1.263013	.124614	.345216	.249856
	7.5	.022653	.304669	.896671	.109017
	8.0	.143189	.029764	.174864	.085872

NOTE.—This table summarizes the regression statistics of the regression between the option prices calculated based on the model and the market price of the option using the following regression equation: Market Price = $a + b$ (Model Price) + error. The option prices were calculated under the assumption of the lognormal distribution at the expiry of the option.

TABLE 7 **Regression Statistics, Options with Weibull Distribution: Model vs. Market Price**

Tenor	Rate (%)	*a*	*b*	R^2	SE
2-year	5.0	15.8327	.3119	.03012	11.1834
	5.5	12.7737	.1940	.01811	5.4759
	6.0	7.5497	.2184	.02776	2.8200
	6.5	.9439	.6867	.01690	1.5398
	7.0	1.8395	.3356	.11415	.6621
	7.5	.1516	.4036	.00761	.2929
3-year	5.0	5.8884	.6706	.11066	10.2994
	5.5	12.8212	.1779	.01648	5.4913
	6.0	7.6806	.1781	.01918	2.8322
	6.5	1.0277	.6961	.01428	1.6769
	7.0	1.8678	.2271	.08134	.5192
	7.5	.1726	.2142	.00170	.3134
4-year	5.0	.9499	.8392	.36618	4.7312
	5.5	10.7210	.3136	.03295	6.8113
	6.0	7.5159	.1647	.02180	2.7538
	6.5	.9128	.8214	.05244	1.6776
	7.0	1.8171	.1780	.07469	.4372
	7.5	.1764	.4383	.02309	.3197
5-year	5.5	6.4987	.5791	.06171	7.4441
	6.0	7.4130	.1529	.01901	2.7091
	6.5	.8842	.8138	.05335	1.6771
	7.0	1.7988	.1472	.06562	.3573
	7.5	.1899	.6177	.03778	.3240
	8.0	.2577	.2222	.10756	.1051
6-year	5.5	.7098	.9447	.80429	2.8985
	6.0	3.1581	.6313	.57707	1.8058
	6.5	−.0005	1.0324	.99982	.0233
	7.0	1.1980	.3331	.38971	.2656
	7.5	−.0009	.8898	.98144	.0454
	8.0	.1059	.4026	.52841	.0687
7-year	5.5	−.0001	.9902	.99999	.0113
	6.0	1.0216	.8849	.85495	1.1214
	6.5	−.0008	1.0333	.99977	.0262
	7.0	.9413	.4350	.44510	.2300
	7.5	−.0003	.8942	.98260	.0447
	8.0	.2101	.0540	.04229	.0925

NOTE.—This table summarizes the regression statistics of the regression between the option prices calculated based on the model and the market price of the option using the following regression equation: Market Price = $a + b$ (Model Price) + error. The option prices were calculated under the assumption of the Weibull distribution at the expiry of the option.

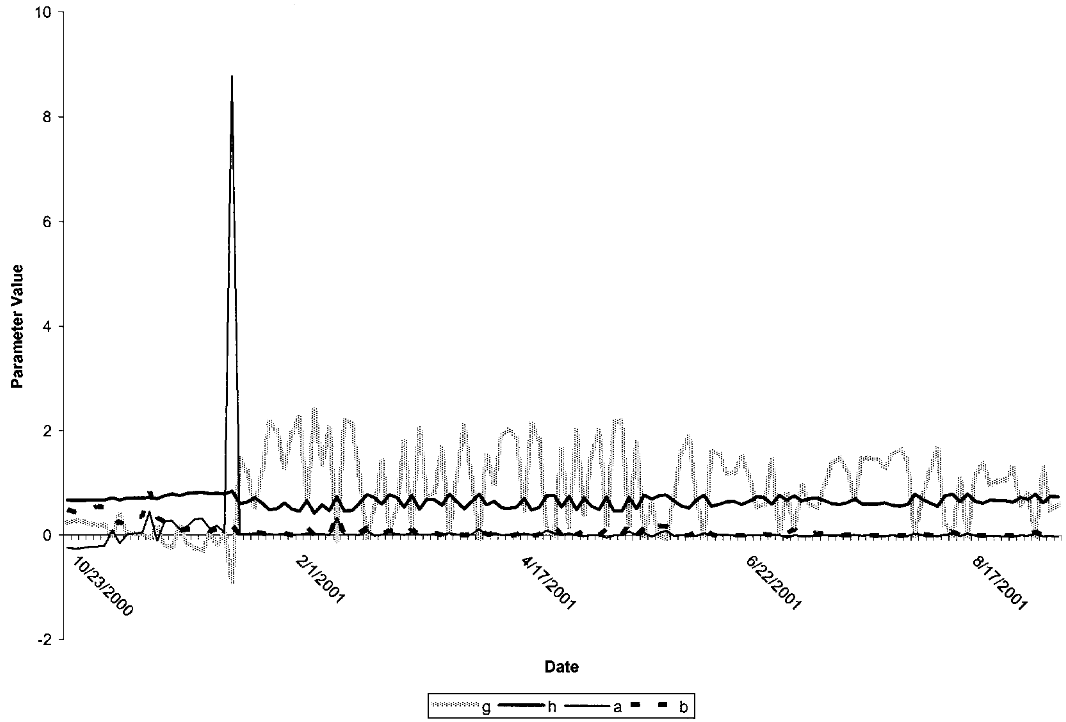


FIG. 2.—Implied parameter estimates for the g-and-h distribution. The figure shows the average estimates of the parameter values across all maturities.

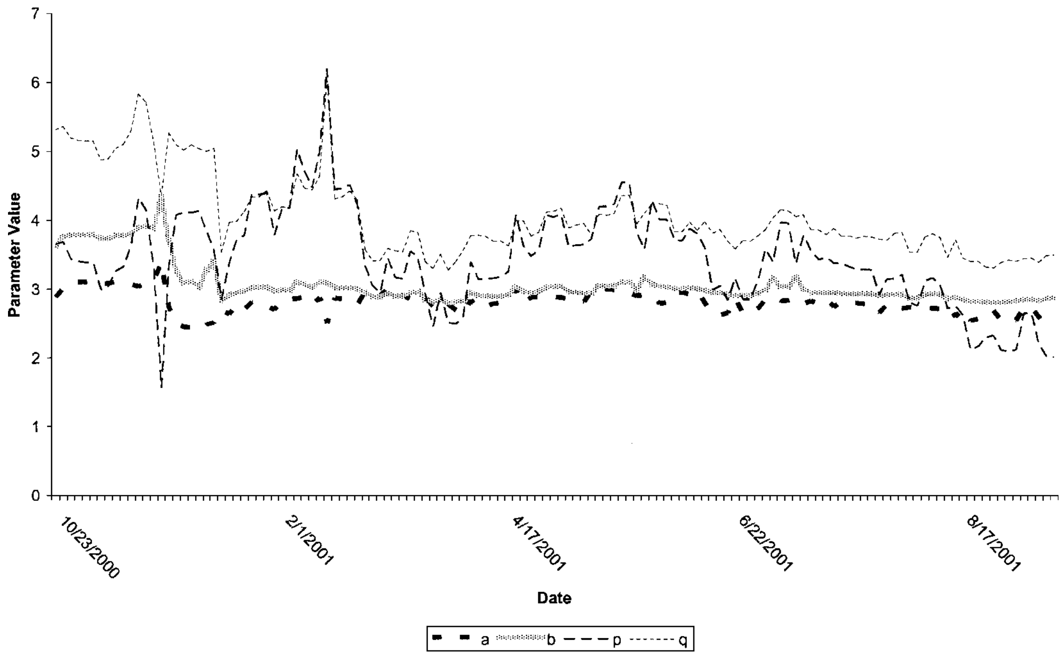


FIG. 3.—Implied parameter estimates for the GB2 distribution. The figure shows the average estimates of the parameter values across all maturities.

TABLE 8 Summary of the Correlation of Errors among Distributions

		Model				
Tenor		GB2	g-and-h	Lognormal	Burr-3	Weibull
2-year	GB2	1	.69	−.55	−.69	−.54
	g-and-h		1	−.56	−.43	−.38
	Lognormal			1.0	.84	.63
	Burr-3				1	.58
	Weibull					1
3-year	GB2	1	.59	−.58	−.60	−.78
	g-and-h		1	−.07	−.06	−.31
	Lognormal			1.0	.84	.47
	Burr3				1	.52
	Weibull					1
4-year	GB2	1	.45	−.10	−.91	−.62
	g-and-h		1	.18	−.29	−.16
	Lognormal			1.0	.19	−.13
	Burr3				1	.62
	Weibull					1
5-year	GB2	1	.59	.86	−.54	−.76
	g-and-h		1	.50	−.12	−.47
	Lognormal			1.0	−.62	−.76
	Burr-3				1	.36
	Weibull					1
6-year	GB2	1	.65	.84	−.57	−.21
	g-and-h		1	.56	−.18	−.16
	Lognormal			1.0	−.59	−.24
	Burr-3				1	.21
	Weibull					1
7-year	GB2	1	.45	.70	−.40	.11
	g-and-h		1	.45	−.12	.15
	Lognormal			1.0	−.44	.05
	Burr-3				1	.04
	Weibull					1

NOTE.—This table presents the summary of the correlation of errors (market_price − model price) among the distributions we use to estimate the implied distribution. The time series of the error is computed by adding the errors for all the options of different strikes on a given day.

between different distributional assumptions. From table 8, we can see that there is positive correlation between the g-and-h and GB2 distributions. The highest and lowest correlation coefficients between these two distributions are 0.69 and 0.45, respectively. With respect to the g-and-h distribution, we observe virtually no correlations with the Burr-3 and Weibull distributions. For shorter maturities, the g-and-h distribution shows negative correlation with the lognormal distribution, but at longer maturities, the correlation coefficient is positive and of approximately the same value as with the GB2.

V. Conclusion

Dutta and Babbel (2002) observed that the skewed and leptokurtic behavior of LIBOR could be modeled effectively by the g-and-h distribution.

The estimates they made can be viewed as backward looking since it was based on what actually happened in the past. The market's expectation of the distributional properties of LIBOR can be extracted from option prices. Here, we attempted to model the skewed and leptokurtic behavior of the 3-month LIBOR data as implied by its option prices. In that respect, the estimates made here could be thought of as forward looking. We observed that the implied distribution of 3-month LIBOR could be modeled very accurately with the g-and-h distribution. Gupta and Subrahmanyam (2001) priced U.S. dollar caps using many well-known term structure models and reported errors in many instances of a higher magnitude than what we obtained using the simple g-and-h distribution. In addition, the regression statistics along with the correlation of errors with other distributions signify an extremely good fit between the implied distribution of the 3-month LIBOR data and the g-and-h distribution. Therefore, we can conclude that the market expected 3-month LIBOR to be skewed and leptokurtic, which can be modeled by the g-and-h distribution with a high degree of accuracy.

The GB2 distribution is also a general skewed and leptokurtic distribution, and we have every reason to believe that we could have modeled the implied distribution with the GB2 just as accurately as with the g-and-h distribution. The inaccuracy we observed in GB2-based prices is probably due to the complexity involved in computing such prices, as is evident from (19). Dutta and Babbel (2002) observed that the GB2 distribution is highly sensitive to its parameter values. Small changes in the parameter values may result in large differences in the option prices. The computational simplicity of the g-and-h distribution is definitely one of the reasons for the accuracy we observed in its prices over the GB2 distribution. Rebonato (1999) reported very high degrees of accuracy in cap (caplet) prices of the DEM caplets using GB2. However, he reported the result for only one trading day.[14]

Even though some authors reported great success in modeling skewness and kurtosis by Burr-3 and Weibull, we did not observe a good fit for our application. These distributions with a restricted number of free parameters could not model the skewed and leptokurtic behavior of the 3-month LIBOR effectively. Based on the statistics observed, we conclude that the option implied distribution of the 3-month LIBOR is not lognormal either. The ample data on 3-month LIBOR options led us to focus on that tenor and instrument in our experiments. Neither in our development of the model nor in our testing did we assume any particular economic properties of 3-month-LIBOR. Therefore, we strongly believe that other short-rates can also be modeled effectively by the g-and-h distribution.

14. Rebonato (1999) claimed to have obtained similar results for many other trading days. It is not clear if the experiment was conducted for substantially longer periods like ours.

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