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Generalized Put–Call Parity

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ABSTRACT

The standard put-call parity result does not include equalities based on buy-and-hold strategies for options on the minimum or maximum of two risky assets and for quantity-adjusting options. This article generalizes put-call parity to these contracts. International put-call parity relations and the pricing of a new forward contract, an absolute-value spread forward, is derived from the put-call parity generalization to options on the minimum or maximum of two risky assets. Finally, an inequality comparing the price of quantity-adjusting options to portfolios of standard options is presented, showing that the QAO contract, in the absence of arbitrage opportunities, is cheaper than the portfolio of standard currency and equity contracts which might be used to hedge the domestic value of a foreign portfolio.

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Put-call parity has been a standard result in the finance literature for over 20 years (Stoll 1969). Curiously, since Stoll's result, no generalizations of put-call parity have appeared. We say "curiously" because many types of options contracts have been subject to study in the last 20 years, yet straightforward generalizations of put-call parity to some of these contracts have not appeared in this literature. Two such option contracts are options on the minimum or maximum of two risky assets (Stulz 1982) and quantity-adjusting options (Babbel and Eisenberg 1991). This paper seeks to fill this gap and generalizes the now-standard result of Stoll to these contracts.[1] It also takes note of symmetries in put-call parity that have not been noted previously in the literature.

In Section I, put-call parity is generalized to options on the minimum or maximum of two assets. Section II uses the results of Section I to extend put-call parity to quantity-adjusting options. Section III presents international put-call parity relationships. Section IV applies the put-call results on options on the minimum or maximum of two risky assets to a new contract called an **absolute value spread forward**. Section V presents an inequality relating quantity-adjusting options to nonquantity-adjusting options. Section VI presents more general inequalities relating product options (POs), quantity-adjusting options (QAOs), and quantity-adjusting forward contracts (QAFs). Section VII concludes our study.

I. PUT-CALL PARITY FOR OPTIONS ON THE MINIMUM OR MAXIMUM OF TWO RISKY ASSETS

The standard put-call parity result states that for European options expiring on the same date, T, with the same exercise price, K, on the same underlying security and with no dividends or coupon payments:

$$C(t) - P(t) + PV(t, K) = S(t) \qquad (1.1)$$

where C(t) is the value of the call at time t, P(t) is the value of the put, PV(t, K) is the present value of the strike price, and S(t) is the price of the underlying security. The purpose of this section is to generalize equation 1.1 for options on the minimum or maximum of two risky assets. Note that equation 1.1 can also be written at time T as:

$$\max\{0, S - K\} - \max\{0, K - S\} + K = S \qquad (1.2)$$

More generally:

$$\max\{x, y\} - \max\{-x, -y\} = x + y \qquad (1.3)$$

where equation 1.3 specializes to equation 1.2 if:

$$x = 0;\ y = S - K \tag{1.4}$$

Equation 1.3 holds because:

$$\text{(i)}\quad x > y:\ \max\{x, y\} = x \tag{1.5}$$
$$-\max\{-x, -y\} = y$$

$$\text{(ii)}\quad x < y:\ \max\{x, y\} = y \tag{1.6}$$
$$-\max\{-x, -y\} = x$$

$$\text{(iii)}\quad x = y:\ \max\{x, y\} = x = y \tag{1.7}$$
$$-\max\{-x, -y\} = x = y$$

We will proceed in this section using:

$$m(x, y) \equiv \min\{x, y\} \tag{1.8}$$

$$M(x, y) \equiv \max\{x, y\} \tag{1.9}$$

In this notation, put-call parity would be

$$M(x, y) - M(-x, -y) = x + y \tag{1.10}$$

where x and y are given by equation 1.4. By reasoning similar to that which justifies equation 1.3 and summarizing:[2]

$$M(x, y) + m(-x, -y) = 0 \tag{1.11}$$

$$m(x, y) + M(-x, -y) = 0 \tag{1.12}$$

$$M(x, y) + m(x, y) = x + y \tag{1.13}$$

$$M(x, y) - m(x, y) = |x - y| \tag{1.14}$$

$$m(x, y) - m(-x, -y) = x + y \tag{1.15}$$

$$M(x, y) - M(-x, -y) = x + y \tag{1.16}$$

$$M(x, y) + M(-x, -y) = |x - y| \tag{1.17}$$

With the exception of equations 1.14 and 1.17, the right-hand sides of these equations represent taking a position in the risky assets x and y. Equations 1.14 and 1.17 require the existence of a forward contract which pays $|x - y|$ on the delivery date. In Section IV, however, we show that this contract is also spanned by the contract which pays M(x, y) and the underlying securities x and y.

To see how one of equations 1.11-1.17 compares to standard put-call parity, note that just as equation 1.15 is the standard result when x and y are given by equation 1.4, equation 1.14 using those values gives:

$$\max\{0, S - K\} - \min\{0, S - K\} = |S - K| \tag{1.18}$$

Note that, by the same reasoning, the original put-call parity results (equations 1.11-1.17) hold for not only the expiration date, but for any date prior for European style options. Hence, for example, M(x, y) can be interpreted not only as a payoff on the expiration date, but more generally as a contract price both prior to and at expiration of a contract whose payoff at expiration is M(x, y).

II. PUT-CALL PARITY FOR QUANTITY-ADJUSTING OPTIONS

Two- and Three-Contract Spreads

When x and y are given by equation 1.4, the equations 1.11-1.17 are arbitrage results with buy and hold strategies for nonquantity-adjusting options. The next step is to extend equations 1.11-1.17 to POs (the general case of quantity-adjusting options) which were presented in Babbel and Eisenberg (1991).[3] First, the extension is made for two-contract spreads where the contracts are POs. Analogous to equations 1.11-1.17, there are seven equations 2.1-2.7. In any specific equation, E(u, v) is equal to either M(u, v) or m(u, v) but not both. The two-contract PO equalities are:

$$M(x, y)\ E(u, v) + m(-x, -y)\ E(u, v) = 0 \tag{2.1}$$

$$m(x, y)\ E(u, v) + M(-x, -y)\ E(u, v) = 0 \tag{2.2}$$

$$M(x, y)\ E(u, v) + m(x, y)\ E(u, v) = (x + y)\ E(u, v) \tag{2.3}$$

$$M(x, y)\ E(u, v) - m(x, y)\ E(u, v) = |x - y|\ E(u, v) \tag{2.4}$$

$$m(x, y)\ E(u, v) - m(-x, -y)\ E(u, v) = (x + y)\ E(u, v) \tag{2.5}$$

$$M(x, y)\, E(u, v) - M(-x, -y)\, E(u, v) = (x + y)\, E(u, v) \tag{2.6}$$

$$M(x, y)\, E(u, v) + M(-x, -y)\, E(u, v) = |x - y|\, E(u, v) \tag{2.7}$$

As with previous comments regarding put-call parity for options on the minimum or maximum of two risky assets for equations 1.14 and 1.17, here equations 2.4 and 2.7 require the existence of a quantity-adjusting option whose payoff is $|x - y|\, E(u, v)$.

Four- and Five-Contract Spreads

The next step is to extend PO put-call parity results for four- and five-contract PO spreads. These equations are the "cross products" of equations 1.11-1.17. Each equation of the 49 cross-product equations is made by writing each of equations 2.1-2.7 in (x, y) and multiplying by each of equations 2.1-2.7 written in (v, u). The equations are listed in Appendix A. To illustrate, consider equation 1.14 crossed with equation 1.16:

$$\begin{aligned} &[M(x, y) - m(x, y)][M(u, v) - M(-u, -v)] \\ &= M(x, y)\, M(u, v) - m(x, y)\, M(u, v) \\ &- M(x, y)\, M(-u, -v) + m(x, y)\, M(-u, -v) \\ &= |x - y|\, (u + v) \end{aligned} \tag{2.8}$$

To compare equation 2.8 with the familiar put-call parity result let:

$$\begin{aligned} x &= 0 \\ y &= S - K \\ u &= 0 \\ v &= X - K_X \end{aligned} \tag{2.9}$$

where S is (extending endnote 3) the Japanese stock price in yen, K is the strike price in yen, X is the spot dollar-yen exchange rate, and K_X is the strike exchange rate. Equation 2.8 becomes:

$$C(S, K)\, C(X, K_X) - \min\{S - K, 0\}\, C(X, K_X) \tag{2.10}$$

$$- C(S, K)\, P(X, K_X) + \min\{0, S - K\}\, P(X, K_X)$$

$$= |K - S|\, (X - K_X)$$

The appendix has all of the equations for five-contract spreads.

III. INTERNATIONAL PUT-CALL PARITY

Babbel and Eisenberg (1991) presented a model for QAOs. Such contracts include a European call option on the Nikkei 225, cash settled in Japanese yen, and converted into U.S. dollars at an exchange rate fixed at the date of issue of the warrant. From equation 2.6, set v to zero and u to a constant exchange rate. Also set y to zero and x to the yen price of the Nikkei 225 minus the yen strike price:

$$v = 0$$

$$u = u^{*}\left(\frac{\$}{¥}\right) \tag{3.1}$$

$$y = 0$$

$$x = n(¥) - k(¥)$$

This gives:

$$M(n - k, 0)\, u^{*} - M(k - n, 0)u^{*} = (n - k)u^{*} \tag{3.2}$$

which is international put-call parity applied to QAOs. Similarly, if instead of setting v to zero, v is set to the spot dollar/yen exchange rate on expiration, then one obtains:

$$M(n - k, 0)\, M(v, u^{*}) - M(k - n, 0)\, M(v, u^{*}) \tag{3.3}$$

$$= (n - k)\, M(v, u^{*})$$

which is the international put-call parity relation for QAOs with an optional exchange rate.

In fact, any generalization of put-call parity in equations A.18–A.59 where (u, v) have dimensions of exchange rates, can be interpreted as an instance of international put-call parity. For example, consider a PO contract which allows U.S. dollar-based investors to convert the Japanese yen payoff to an option on the maximum of two Japanese stock indices, x and y, into deutsche marks or Swiss francs at fixed exchange rates $v^*\left(\frac{SFr}{¥}\right)$ or $u^*\left(\frac{DM}{¥}\right)$, which is the maximum of these quantities valued in U.S. dollars at the spot rate on expiration. Let $a\left(\frac{\$}{SFr}\right)$ be the U.S. dollar-Swiss franc spot rate and $b\left(\frac{\$}{DM}\right)$ be the dollar-deutsche mark spot rate. Then international put-call parity can be applied to a contract which pays M(x, y) M(av*, bu*). This contract with other suitable contracts can be used in equations A.18-A.59. All of these can be interpreted as instances of international put-call parity.

IV. ABSOLUTE-VALUE SPREAD FORWARDS

Note that a portfolio of an option on the minimum or maximum of shares of two risky assets E(x, y), and short a half share of each asset is an absolute-value spread forward:

$$
\begin{aligned}
M(x, y) - \frac{x+y}{2} &= M\left(\frac{x-y}{2}, \frac{y-x}{2}\right) \\
&= \frac{1}{2}\,|x-y|
\end{aligned}
\tag{4.1}
$$

and

$$
m(x, y) - \frac{x+y}{2} = -\frac{1}{2}\,|x-y| \tag{4.2}
$$

Thus, an absolute-value spread forward is duplicated with a buy-and-hold strategy that is long two maximum options on a share each of two risky assets and short the risky assets. Hence, to carry out the arbitrage in equation 1.14 where, say, the market price permits arbitrage because the left-hand side is priced higher than the right-hand side, means that:

$$
M_t(x, y) - m_t(x, y) > 2M_t(x, y) - (x_t + y_t) \tag{4.3}
$$

Hence, $M_t(\cdot)$ and $m_t(\cdot)$ are taken to denote the option prices at time t, and x_t and y_t denote the prices of the underlying, also at t. Note that in the absence of arbitrage opportunities, the inequality sign in equation 4.3 would be replaced by an equality. Note also that equation 4.3 is obtained by subtracting equation 4.2 from equation 4.1 and equating the result with twice equation 4.1. Assuming an arbitrage opportunity as presented by inequality in equation 4.3, one obtains:

$$M_t(x, y) + m_t(x, y) < x_t + y_t \tag{4.4}$$

Note that in the absence of a forward contract which pays $|x - y|$, equations 1.13 and 1.14 represent the same arbitrage.

V. AN INEQUALITY: HYBRID PUT VERSUS STANDARD PUTS

This section shows that in the absence of arbitrage opportunities, a hybrid put option, Z(t), on a non-domestic equity portfolio is as cheap or cheaper than a portfolio using standard equity and foreign currency put options.

For example, suppose a U.S. dollar-based investor owns a Japanese equity portfolio and wants to protect the U.S. dollar value of that portfolio at some arbitrary U.S. dollar level noted by G(\$). Decompose G(\$) into the product of any two arbitrary numbers with dimensions Japanese yen and U.S. dollars/Japanese yen, respectively:

$$G(\$) = K(¥) \cdot K_X\left(\frac{\$}{¥}\right) \tag{5.1}$$

Let K(¥) be the exercise price of a standard put option on the Japanese stock portfolio, and $K_X\left(\frac{\$}{¥}\right)$ be the strike price of K put options on Japanese yen to sell 1 yen for at least X(0, T)(\$).

In the absence of arbitrage opportunities, the dollar-based investor will find the cost of this portfolio of K currency put options and one standard yen-denominated equity put option greater than or equal to the cost of a hybrid put option:[4]

$$Z(T) = \max\{X(T), K_X\}\left(\frac{\$}{¥}\right) \cdot \max\{0, K - N(T)\}(¥) \tag{5.2}$$

where:

$$T \equiv \text{expiration date of the hybrid option} \tag{5.3}$$

$$N(T)(¥) \equiv \text{price of Japanese stock portfolio in Japanese yen}$$

$$X(T)\left(\frac{\$}{¥}\right) \equiv \text{the spot dollar-yen exchange rate at T}$$

$$Z(T) = [\max\{0, K_X - X(T)\} + X(T)] \bullet \max\{0, K - N(T)\} \tag{5.4}$$

$$Z(T) = \max\{0, K_X - X(T)\} \bullet \max\{0, K - N(T)\} + X(T)\max\{0, K - N(T)\} \tag{5.5}$$

Since:

$$\max\{0, K - N(T)\} \le K \tag{5.6}$$

$$Z(T) \le \max\{0, K_X - X(T)\} \bullet K + X(T) \bullet \max\{0, K - N(T)\} \tag{5.7}$$

Note that strict inequality holds in equation 5.7 if $N(T) > 0$. The first term on the right-hand side is the payoff at time T in U.S. dollars of a put option on K Japanese yen struck at K_X dollars per yen. The second term on the right-hand side is the payoff of a standard put option on the Japanese stock portfolio struck at K Japanese yen and valued in U.S. dollars on expiration at the spot rate.

Note that the hedge with the portfolio of "vanilla" options on currency and foreign stock, respectively, does not provide an exact hedge of currency translation risk because the currency options are on a fixed amount of foreign currency. This fixed amount need not be equal to the yen payoff of the Nikkei 225 put.

Note also that the dominance result in this section is *not* a version of the theorem that a portfolio of options is worth more than an option on a portfolio (Merton 1973). In the result here, instead of an option on a portfolio, we have a quantity-adjusting option, and instead of a portfolio, all with options on stock, we have a portfolio of options on stock and foreign currency.

VI. MORE GENERAL COMPARISONS OF DOMINANCE RELATIONS BETWEEN DIFFERENT TYPES OF OPTIONS

The previous section demonstrated that in the absence of arbitrage opportunities, a QAO put on, say, a Japanese stock is cheaper than a portfolio made up of an at-the-money "vanilla" put on the stock (denominated in Japanese yen) and a vanilla currency put to sell yen into the investor's home currency, say U.S. dollars, where the currency put is on a quantity of yen equal to the current yen price of the Japanese stock. In Part A of this section, we will consider a more general equation and consequent inequality that will allow us to compare not only QAOs to standard options (options on one risky asset), but also POs to options on the minimum or maximum of two risky assets and to QAOs. In Parts B, C, and D, three other inequalities are given. The first compares the prices of POs to QAOs. The second and third inequalities compare prices of QAOs to QAFs, and POs to QAFs, respectively.

Inequalities Relating POs, QAOs, and QAFs

From equation 4.1:

$$M(x, y) - \frac{x + y}{2} = \frac{1}{2}|x - y| \tag{6.1}$$

$$M(u, v) - \frac{u + v}{2} = \frac{1}{2}|u - v|$$

Hence:

$$\begin{aligned} &M(x, y)M(u, v) - \frac{1}{2}(x + y)M(u, v) - \frac{1}{2}(u + v)M(x, y) \\ &+ \frac{1}{4}(x + y)(u + v) = \frac{1}{4}|x - y|\ |u - v| \geq \frac{\psi}{4} \end{aligned} \tag{6.2}$$

$$\psi = (x - y)(u - v),\ (y - x)(u - v),\ (x - y)(v - u),\ (y - x)(v - u) \tag{6.3}$$

Taking the first of the four cases for ψ in equation 6.3, then equation 6.2 becomes:

$$\begin{aligned} &M(x, y)M(u, v) - \frac{1}{2}(x + y)M(u, v) - \frac{1}{2}(u + v)M(x, y) \\ &+ \frac{1}{4}(xu + xv + yu + yv) \\ &\geq \frac{1}{4}(xu - xv - yu + yv) \end{aligned} \tag{6.4}$$

Using ϕ as defined below, this simplifies to:

$$M(x, y)M(u, v) - \frac{1}{2}(x + y)M(u, v) - \frac{1}{2}(u + v)M(x, y) \geq -\frac{\phi}{2}\ ;\ \phi = (xv - yu) \tag{6.5}$$

The following additional (ψ, ϕ) pairs obtain:

$$\psi = (y - x)(u - v);\ \ \phi = yv + xu \tag{6.6}$$

$$\psi = (x - y)(v - u);\ \ \phi = yv + yu \tag{6.7}$$

$$\psi = (y - x)(v - u);\ \ \phi = xv + yu \tag{6.8}$$

Equations 6.5-6.8 give dominance relations between POs, QAOs, and QAFs. Note that in equation 6.5 the first term is the payoff of a PO at expiration (or more generally, its price prior to T). xM(u, v), yM(u, v), uM(x, y), and vM(x, y) are the same for QAOs. The term on the right-hand side is the sum of two QAFs.

An Inequality for POs and QAOs Only

Another inequality compares the price of a PO directly to that of QAOs independent of the pricing of QAFs:

$$M(x, y)M(u, v) \geq \frac{1}{2}(x + y)M(u, v),\ \frac{1}{2}(u + v)M(x, y) \tag{6.9}$$

Equation 6.9 is obtained as follows. Note that:

$$M(x, y) \geq x, y \tag{6.10}$$

$$2M(x, y) \geq x + y \tag{6.11}$$

$$M(x, y) \geq \frac{x + y}{2} \tag{6.12}$$

$$M(u, v) \geq \frac{u + v}{2} \tag{6.13}$$

Multiplying equations 6.12 and 6.13, respectively, by M(u, v) and M(x, y) yields equation 6.9.

An Inequality for QAOs and QAFs Only

By equations 6.10 and 6.13:

$$(u + v)M(x, y),\ (x + y)M(u, v) \geq \frac{1}{2}(x + y)(u + v) \tag{6.14}$$

An Inequality for POs and QAFs Only

By equations 6.9 and 6.14:

$$M(x, y)M(u, v) \geq \frac{1}{4}(x + y)(u + v) \tag{6.15}$$

VII. CONCLUSION

Buy and hold trading strategies such as those represented by put-call parity are robust to different characterizations of the stochastic process of the returns on the underlying security. In periods when traders may be unwilling to rely on a dynamic strategy dependent upon a characterization of the stochastic return processes of the underlying risky assets (such as on October 19, 1987), buy-and-hold strategies can be useful both to arbitrage existing markets and to create new markets because such strategies are robust to assumptions about return processes.

Since put-call parity was first represented in the literature, many types of options contracts have been discussed. This article has generalized the standard result to options on the minimum or maximum of two assets and to quantity-adjusting options with applications to international put-call parity.

APPENDIX: GENERALIZED PUT-CALL PARITY RESULTS

Two-Option Spread Equalities for Nonquantity-Adjusting Options

$$M(x, y) + m(-x, -y) = 0 \tag{A.1}$$

$$m(x, y) + M(-x, -y) = 0 \tag{A.2}$$

$$M(x, y) + m(x, y) = x + y \tag{A.3}$$

$$M(x, y) - m(x, y) = |x - y| \tag{A.4}$$

$$m(x, y) - m(-x, -y) = x + y \tag{A.5}$$

$$M(x, y) - M(-x, -y) = x + y \tag{A.6}$$

$$M(x, y) + M(-x, -y) = |x - y| \tag{A.7}$$

Two-Option Equalities for Nonquantity-Adjusting Options in Standard Put and Call Notation

Using the following notation:

$$x = 0; y = S - K \tag{A.8}$$

$$C = \max\{0, S(T) - K)\} \tag{A.9}$$

$$P = \max \{0, K - S(T)\} \tag{A.10}$$

Equations A.1-A.7 become respectively:

$$C + \min\{0, K - S\} = 0 \tag{A.11}$$

$$\min \{0, S - K\} + P = 0 \tag{A.12}$$

$$C + \min\{0, S - K\} = S - K \tag{A.13}$$

$$C - \min\{0, S - K\} = |K - S| \tag{A.14}$$

$$\min\{0, S - K\} - \min\{0, K - S\} = S - K \tag{A.15}$$

$$C - P = S - K \text{ (put-call parity)} \tag{A.16}$$

$$C + P = |K - S| \tag{A.17}$$

Three-Option PO Spread Equalities (Includes Nonquantity-Adjusting Options)

In any of the equations in this section, E(u, v) should be interpreted as either m(u, v) or M(u, v). Equations A.1-A.7 become respectively:

$$M(x, y)\, E(u, v) + m(-x, -y)\, E(u, v) = 0 \tag{A.18}$$

$$m(x, y)\, E(u, v) + M(-x, -y)\, E(u, v) = 0 \tag{A.19}$$

$$M(x, y)\, E(u, v) + m(x, y)\, E(u, v) = (x + y)\, E\, (u, v) \tag{A.20}$$

$$M(x, y)\, E(u, v) - m(x, y)\, E(u, v) = |x - y|\, E(u, v) \tag{A.21}$$

$$m(x, y)\, E(u, v) - m(-x, -y)\, E(u, v) = (x + y)\, E(u, v) \tag{A.22}$$

$$M(x, y)\, E(u, v) - M(-x, -y)\, E(u, v) = (x + y)\, E(u, v) \tag{A.23}$$

$$M(x, y)\, E(u, v) + M(-x, -y)\, E(u, v) = |x - y|\, E(u, v) \tag{A.24}$$

Four-Option PO-Spread Equalities in Standard Put and Call Notation

Along with equations A.8-A.10, let:

$$u = 0;\ v = X - K_X \tag{A.25}$$

obtaining:

$$C\, E(u, v) + \min\{0, K - S\}\, E(u, v) = 0 \tag{A.26}$$

$$\min\{0, S - K\}\, E(u, v) + P\, E(u, v) = 0 \tag{A.27}$$

$$C\, E(u, v) + \min\{0, S - K\}\, E(u, v) = (S - K)\, E(u, v) \tag{A.28}$$

$$C\, E(u, v) - \min\{0, S - K\}\, E(u, v) = |K - S|\, E(u, v) \tag{A.29}$$

$$\min\{0, S - K\}\, E(u, v) - \min\{0, K - S\}\, E(u, v) = (S - K)\, E(u, v) \tag{A.30}$$

$$C\, E(u, v) + \bar{P}\, E(u, v) = |K - S|\, E(u, v) \tag{A.31}$$

Five-Option PO-Spread Equalities

Each of the following equations may be thought of as corresponding to a cell of a 7 X 7 matrix. Only the upper triangular portion of the matrix is given. Equalities corresponding to a lower triangular cell (i, j) may be obtained from the equation corresponding to the upper triangular cell (j, i) and switching each occurrence of x with that of u and each occurrence of y with v. Table 1 gives the equation numbers corresponding to the upper-triangular cells.

Table 1
Four-Option PO Spread Equalities

	A.1	A.2	A.3	A.4	A.5	A.6	A.7
A.1	11	12	13	14	15	16	17
A.2		22	23	24	25	26	27
A.3			33	34	35	36	37
A.4				44	45	46	47
A.5					55	56	57
A.6						66	67
A.7							77

A.1 Equalities

Going from cell 11 to cell 17 in the matrix gives seven equalities:

$$
\begin{aligned}
&[M(x, y) + m(-x, -y)][M(u, v) + m(-u, -v)] \\
&= M(x, y)M(u, v) + m(-x, -y)M(u, v) \\
&+ M(x, y)m(-u, -v) + m(-x, -y)m(-u, -v) \\
&= 0
\end{aligned} \tag{A.32}
$$

$$
\begin{aligned}
&[M(x, y) + m(-x, -y)][m(u, v) + M(-u, -v)] \\
&= M(x, y)m(u, v) + m(-x, -y)m(u, v) \\
&+ M(x, y)M(-u, -v) + m(-x, -y)M(-u, -v) \\
&= 0
\end{aligned} \tag{A.33}
$$

$$
\begin{aligned}
&[M(x, y) + m(-x, -y)][M(u, v) + m(u, v)] \\
&= M(x, y)M(u, v) + m(-x, -y)M(u, v) \\
&+ M(x, y)m(u, v) + m(-x, -y)m(u, v) \\
&= 0
\end{aligned} \tag{A.34}
$$

$$
\begin{aligned}
&[M(x, y) + m(-x, -y)][M(u, v) - m(u, v)] \\
&= M(x, y)M(u, v) + m(-x, -y)M(u, v) \\
&+ m(-x, -y)M(u, v) - m(-x, -y)m(u, v) \\
&= 0
\end{aligned} \tag{A.35}
$$

$$[M(x, y) + m(-x, -y)][m(u, v) - m(-u, -v)] \tag{A.36}$$
$$= M(x, y)m(u, v) + m(-x, -y)m(u, v)$$
$$+ m(-x, -y)m(u, v) - m(-x, -y)m(-u, -v)$$
$$= 0$$

$$[M(x, y)m(-x, -y)][M(u, v) - M(-u, -v)] \tag{A.37}$$
$$= M(x, y)M(u, v) + m(-x, -y)M(u, v)$$
$$- M(x, y)M(-u, -v) - m(-x, -y)M(-u, -v)$$
$$= 0.$$

$$[M(x, y) + m(-x, -y)][M(u, v) + M(-u, -v)] \tag{A.38}$$
$$= M(x, y)M(u, v) + m(-x, -y)M(u, v)$$
$$+ M(x, y)M(-u, -v) + m(-x, -y)M(-u, -v)$$
$$= 0$$

A.2 Equalities

Going from cell 22 to cell 27 gives six equalities:

$$[m(x, y) + M(-x, -y)][m(u, v) + M(-u, -v)] \tag{A.39}$$
$$= m(x, y)m(u, v) + M(-x, -y)m(u, v)$$
$$+ m(x, y)M(-u, -v) + M(-x, -y)M(-u,-v)$$
$$= 0$$

$$[m(x, y) + M(-x, -y)][M(u, v) + m(u, v)] \tag{A.40}$$
$$= m(x, y)M(u, v) + M(-x, -y)m(u, v)$$
$$+ m(x, y)m(u, v) + M(-x, -y)m(u, v)$$
$$= 0$$

$$[m(x, y) + M(-x, -y)][M(u, v) - m(u, v)] \tag{A.41}$$
$$= m(x, y)M(u, v) + M(-x, -y)M(u, v)$$
$$- m(x, y)m(u, v) - M(-x, -y)m(u, v)$$
$$= 0$$

$$[m(x, y) + M(-x, -y)][m(u, v) - m(-u, -v)] \tag{A.42}$$
$$= m(x, y)m(u, v) + M(-x, -y)m(u, v)$$
$$+ m(x, y)m(-u, -v) - M(-x, -y)m(-u, -v)$$
$$= 0.$$

$$[m(x, y) + M(-x, -y)][M(u, v) - M(-u, -v)] \tag{A.43}$$
$$= m(x, y)M(u, v) - m(x, y)M(-u, -v)$$
$$+ M(-x, -y)M(u, v) - M(-x, -y)M(-u, -v)$$
$$= 0$$

$$[m(x, y) + M(-x, -y)][M(u, v) + M(-u, -v)] \tag{A.44}$$
$$= m(x, y)M(u, v) + M(-x, -y)M(u, v)$$
$$+ m(x, y)M(-u, -v) + M(-x, -y)M(-u, -v)$$
$$= 0$$

A.3 Equalities

Starting with cell 33 and continuing to cell 37 gives five equalities:

$$[M(x, y) + m(x, y)][M(u, v) + m(u, v)] \tag{A.45}$$
$$= M(x, y)M(u, v) + m(x, y)M(u, v)$$
$$+ M(x, y)m(u, v) + m(x, y)m(u, v)$$
$$= (x + y)(u + v)$$

$$[M(x, y) + m(x, y)][M(u, v) - m(u, v)] \tag{A.46}$$
$$= M(x, y)M(u, v) + m(x, y)M(u, v)$$
$$- M(x, y)m(u, v) - m(x, y)m(u, v)$$
$$= (x + y)|u - v|$$

$$[M(x, y) + m(x, y)][m(u, v) - m(-u, -v)] \tag{A.47}$$
$$= M(x, y)m(u, v) + m(x, y)m(u, v)$$
$$- M(x, y)m(-u, -v) - m(x, y)m(-u, -v)$$
$$= (x + y)(u + v)$$

$$[M(x, y) + m(x, y)][M(u, v) - M(-u, -v)] \tag{A.48}$$
$$= M(x, y)M(u, v) + m(x, y)M(u, v)$$
$$- M(x, y)M(-u, -v) - m(x, y)M(-u, -v)$$
$$= (x + y)(u + v)$$

$$[M(x, y) + m(x, y)][M(u, v) + M(-u, -v)] \tag{A.49}$$
$$= M(x, y)M(u, v) + m(x, y)M(u, v)$$
$$+ M(x, y)M(-u, -v) + m(x, y)M(-u, -v)$$
$$= (x + y)|u - v|$$

A.4 Equalities

Starting with cell 44 and continuing to cell 47 gives four equalities:

$$[M(x, y) - m(x, y)][M(u, v) - m(u, v)] \tag{A.50}$$
$$= M(x, y)M(u, v) - m(x, y)M(u, v)$$
$$- M(x, y)m(u, v) + m(x, y)m(u, v)$$
$$= |x - y||u - v|$$

$$[M(x, y) - m(x, y)][m(u, v) - m(-u, -v)] \tag{A.51}$$
$$= M(x, y)m(u, v) - m(x, y)m(u, v)$$
$$- M(x, y)m(-u, -v) + m(x, y)m(-u, -v)$$
$$= |x + y|(u + v)$$

$$[M(x, y) - m(x, y)][M(u, v) - M(-u, -v)] \tag{A.52}$$
$$= M(x, y)M(u, v) - m(x, y)M(u, v)$$
$$- M(x, y)M(-u, -v) + m(x, y)M(-u, -v)$$
$$= |x - y|(u + v)$$

$$[M(x, y) - m(x, y)][M(u, v) + M(-u, -v)] \tag{A.53}$$
$$= M(x, y)M(u, v) - m(x, y)M(u, v)$$
$$+ M(x, y)M(-u, -v) - m(x, y)M(-u, -v)$$
$$= |x - y||u - v|$$

A.5 Equalities

Cell 55 through 57 gives three equalities:

$$
\begin{aligned}
&[m(x, y) - m(-x, -y)][m(u, v) - m(-u, -v)] \\
&= m(x, y)m(u, v) - m(-x, -y)m(u, v) \\
&- m(x, y)m(-u, -v) + m(-x, -y)m(-u, -v) \\
&= (x + y)(u + v)
\end{aligned}
\tag{A.54}
$$

$$
\begin{aligned}
&[m(x, y) - m(-x, -y)][M(u, v) - M(-u, -v)] \\
&= m(x, y)M(u, v) - m(-x, -y)M(u, v) \\
&- m(x, y)M(-u, -v) + m(-x, -y)M(-u, -v) \\
&= (x + y)(u + v)
\end{aligned}
\tag{A.55}
$$

$$
\begin{aligned}
&[m(x, y) - m(-x, -y)][M(u, v) + M(-u, -v)] \\
&= m(x, y)M(u, v) - m(-x, -y)M(u, v) \\
&+ m(x, y)M(-u, -v) - m(-x, -y)M(-u, -v) \\
&= (x + y)|u - v|
\end{aligned}
\tag{A.56}
$$

A.6 Equalities

Cell 66 to cell 67 gives two equalities:

$$
\begin{aligned}
&[M(x, y) - M(-x, -y)][M(u, v) - M(-u, -v)] \\
&= M(x, y)M(u, v) - M(-x, -y)M(u, v) \\
&- M(x, y)M(-u, -v) + M(-x, -y)M(-u, -v) \\
&= (x + y)(u + v)
\end{aligned}
\tag{A.57}
$$

$$
\begin{aligned}
&[M(x, y) - M(-x, -y)][M(u, v) + M(-u, -v)] \\
&= M(x, y)M(u, v) - M(-x, -y)M(u, v) \\
&+ M(x, y)M(-u, -v) - M(-x, -y)M(-u, -v) \\
&= (x + y)|u - v|
\end{aligned}
\tag{A.58}
$$

A.7 Equality

With only the diagonal element left, there is one equation:

$$
\begin{aligned}
&[M(x, y) + M(-x, -y)][M(u, v) + M(-u, -v)] \qquad \text{(A.59)}\\
&= M(x, y)M(u, v) + M(-x, -y)M(u, v)\\
&+ M(x, y)M(-u, -v) + M(-x, -y)M(-u, -v)\\
&= |x - y||u - v|
\end{aligned}
$$

NOTES

1. The authors would like to thank Mark Rubinstein, Robert Jarrow, Nick Valerio, Jack Chang, and Jayant Kale for their comments.

2. The authors would like to thank David Nachman for his comment that equations 1.11–1.17 are an example of a Banach lattice over the reals. See Schaefer (1974), 46-56. While there are 28 possible permutations of $\pm E(\pm x, \pm y) \pm E(\pm x, \pm y)$, we have displayed the eight equations that are of primary interest.

3. POs (product options) are options whose payoff at expiration is the product of the payoffs of two options, where each of the two options is an option on the minimum or maximum of two risky assets. If M(x,y) and M(u,v) are the payoffs to options which are respectively the maximum of x and y and the maximum of u and v, then:

$$Z \equiv M(x,y)\, M(u,v)$$

defines the payoff to a PO. Examples of POs include sequential investment options and (as a special case) quantity-adjusting options. An example of a sequential investment option is an option on the maximum payoff over a two-period investment for two assets with the payoff tied to the sequence of investments which has the highest payoff. An example of a quantity-adjusting option would be an option on the Nikkei 225 stock index with payoff converted from Japanese yen into U.S. dollars at a conversion rate fixed at the outset. In the equation for Z above, let S = Nikkei 225 (¥), $x = 0$, $y = (S - K)$(¥), $u = 0$, $v = v^*$($/¥). Then Z gives the stochastic payoff to a "vanilla" put option on the Nikkei 225 converted into dollars at a fixed exchange rate.

4. Note that the pricing model of the hybrid option is a special case of the model set forth in Babbel and Eisenberg (1991), in Section II, Case (v) of

that paper. By Procedure Two of Section I in that paper, the hybrid option is a special case of Case (v).

REFERENCES

Babbel, D. F. and L. K. Eisenberg, "Quantity-Adjusting Options and Forward Contracts," Rodney White Center Working Paper 24-91, Wharton School, University of Pennsylvania, 1991.

Merton, R., "Theory of Rational Option Pricing," *Journal of Economics and Management Science,* 4, 1973, pp. 141-83.

Schaefer, H. H., *Banach Lattices and Positive Operators*, Springer-Verlag, Berlin, Heidelberg, New York, 1974.

Stoll, H. R., "The Relationships Between Put and Call Options Prices," *The Journal of Finance*, 24, December 1969, pp. 802-24.

Stulz, R. M., "Options on the Minimum or the Maximum of Two Risky Assets: Analysis and Applications," *Journal of Financial Economics,* 10, July 1982, pp. 161-86.