Interest-Rate Option Pricing Revisited

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INTRODUCTION

When valuing derivative securities, if no arbitrage opportunities exist, then the value of the derivative must equal the value of a portfolio of fundamental securities that replicates the payoffs of the contract being valued. The purpose of this note is to point out the care that must be exercised when choosing fundamental securities with which to replicate the cash flows of more complex securities. Often, the choice of fundamental, or replicating, securities is driven by the existence of a known solution for their value.

An example of this approach is Turnbull and Milne (1991), who use the contingent claims framework to derive closed-form solutions for options written on a variety of interest-rate contingent securities. They value each interest-rate contingent security relative to the exogenously given term structure of interest rates. Using this approach, they are able to derive closed-form solutions not only for simple contracts, such as options on discount Treasury bonds, but also for more complex contracts, such as options on interest rate futures contracts.

As shown by Boyle and Turnbull (1989) or Turnbull and Milne (1991) a European put option on a T-bill can be used to replicate the payoff on an interest rate cap. They use this relationship because there are known solutions for the value of a European option on a T-bill. In replicating the interest rate cap, the appropriate number of put options

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on a T-bill must be purchased. In addition, the put options must have a strike price that is related to the strike rate for the interest rate cap. An exact solution for the number and characteristics of the put options is demonstrated by Hull and White (1990). For many interest rate caps it is common to determine the cash flow at the reset date but to delay payment until the end of the reset period just prior to the next reset date. This type of cap, called a deferred cap, is considered by Hull and White. The next section presents their results as a starting point and then compares the results to nondeferred caps.

REPLICATING AN INTEREST RATE CAP

Let R_x be the strike rate for an interest rate cap with notional principal of one dollar and a reset date at time t_1 with the payoff from the cap at time t_2. The payoff at time t_2 will then be

$$\Delta t \max(R_1 - R_x, 0)$$

where R_1 is the interest rate available at time t_1 for the period from time t_1 to time t_2. The interest rates are discrete rates for the period, Δt, stated in annual terms. The discounted value of this payoff is equivalent to

$$(1 + R_x\Delta t) \max\left[\frac{1}{1 + R_x\Delta t} - \frac{1}{1 + R_1\Delta t}, 0\right]$$

at time t_1. As discussed by Hull and White (1990), this expression represents $(1 + R_x\Delta t)$ European put options that expire at time t_1, have a strike price of $1/(1 + R_x\Delta t)$, and are written on a discount bond with face value of one dollar and maturing at time t_2. Because the number of puts to be purchased is known at time $t_0 = t_1 - \Delta t$, the value of the interest rate cap can be replicated at time t_0 by put options on a T-bill.

However, for some interest rate caps the cash flow may occur on the reset date. In this case, it is not possible to replicate the interest rate cap with a put option on a T-bill. The payoff for the interest rate cap will be the same as above, but will occur at time t_1. Using the same process as above, the payoff is equivalent to

$$(1 + R_1\Delta t)(1 + R_x\Delta t) \max\left[\frac{1}{1 + R_x\Delta t} - \frac{1}{1 + R_1\Delta t}, 0\right]$$

at time t_1. Because R_1 will not be known until time t_1, the number of T-bills to hold at time t_0 to replicate this interest rate cap is unknown.

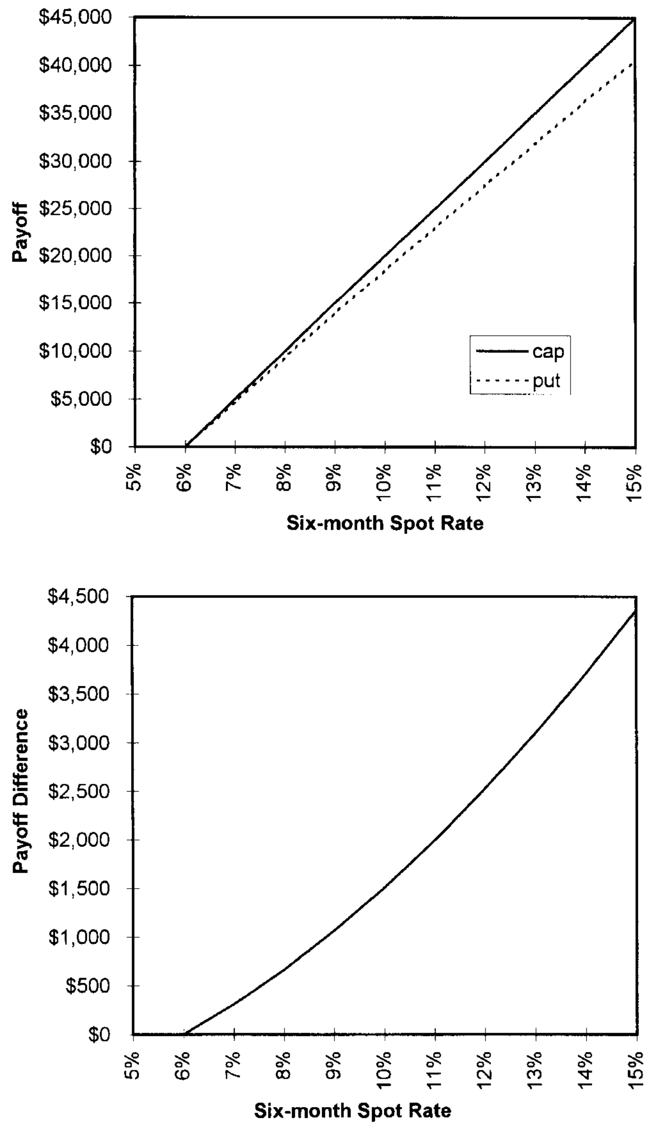

FIGURE 1
(a) Payoff functions. (b) Difference between cap and put option payoffs.

Therefore, the value of this interest rate cap cannot be known at time t_0 by replicating the interest rate cap with put options on a T-bill.

A pricing error that may be introduced when the interest rate cap pays on the reset date can be illustrated by a simple example. Consider a six-month single-pay cap that pays at maturity, time t_1. The cap will pay $(0.5)\max(R_1 - R_x, 0)$ times one million dollars, the notional principal of the contract, where, in this example, R_1 is the level of the six-month spot

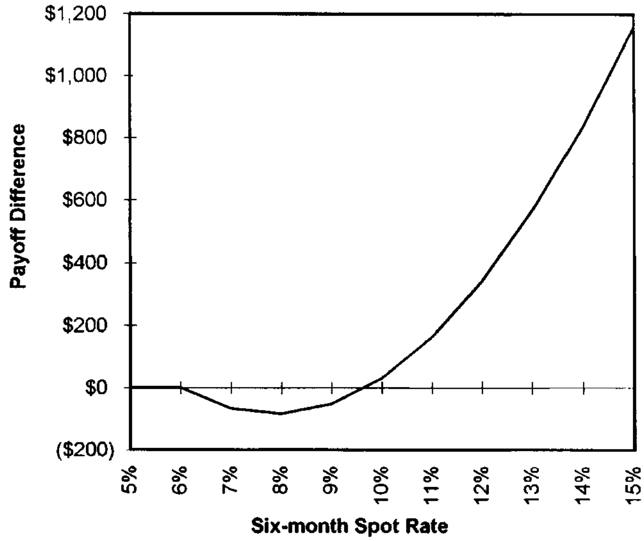

FIGURE 2
Difference between cap and put option payoffs.

rate of interest. Compare this to a six-month put option on a one-year T-bill with face value of $1 million and a strike price of $1/(1 + R_x/2)$.

Figure 1(a) illustrates the payoffs at maturity for both the interest rate cap and the put option on the T-bill. Notice that due to the convexity in the relationship between the value of the bond and the six-month spot rate, the put option underestimates the value of the interest rate cap. The size of the error increases at an increasing rate as the six-month spot rate increases. The replication error is illustrated in Figure 1(b) by the line representing the payoff to the interest rate cap minus the pay-off to the put option on the T-bill.

The sign of the pricing error will depend on the maturity of the underlying T-bill. For example, assuming a flat term structure and a T-bill with 6.5 months to maturity at the expiration of the cap and the put option, the error changes signs. This is illustrated in Figure 2. In general, the longer the remaining maturity of the T-bill at the expiration of the put option, the greater the range of six-month spot rates for which the error is negative.

CONCLUSION

Analysts who rely on the use of options on T-bills to replicate the value of an interest rate cap must be careful to apply it only to deferred caps. Nondeferred caps must be priced using traditional lattice or simulation methods, as there is no closed-form solution for their pricing. Otherwise, a pricing error is introduced due to the nonlinear dependence of the put option value on the spot rate of interest when valuing an instrument with a linear dependence on the spot rate of interest.

A related conclusion is that the choice of hedging instrument is sensitive to the cash flow pattern of the interest rate exposure to be hedged. Options on T-bills or Treasury bonds are not necessarily perfect substitutes for interest rate caps.

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